Raytracing Equations

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1 Introduction

The raytracing method outlined in this document is the same as described in Appendix A of the Interstellar paper by James et al. [1]. We include additional implementation details and clarifications where beneficial.

All rays are traced backwards in time from the position of the camera to the position they originated. As we assume all stars and nebulaes to be infinitely far away from the black hole and the camera, these objects can be projected onto a sphere with infinite radius centered on the black hole. We trace the rays back to this sphere, which we refer to as the celestial sky.

The raytracing steps for every ray can be summarized as follows: first calculate the canonical momenta and the ray's constants of motion. With these parameters, check whether the ray comes from the black hole shadow region or the celestial sky. If the ray comes from the celestial sky, numerically integrate the ray equations 2 backward in time from the camera position to $-\infty$ (or in practice some very large negative number), with as starting parameters the camera position and the computed canonical momenta. Lastly, determine the redshift of the light arriving via the integrated light ray.

1.1 Parameters

The input parameters for the algorithm define the black hole, the observer (camera) and the specific ray to be traced. The black hole is defined by its spin value only, as the mass of the black hole is set to 1 in the equations. For the camera we set the position, movement vector and speed. A ray is specified by its incoming direction at the camera.

With these parameters we compute the ray's constants of motion, which tell us whether or not the ray comes from the celestial sky. Besides the predefined or precomputed parameters, we keep track of the position and canonical momenta of our light ray with every tracing step we take. When we have traced our ray far enough back in time, this position describes the direction our light ray originated from on the celestial sky.

All vectors are defined in spherical coordinates, with the black hole at the origin of the system. The only exception to this is the incoming ray direction, which is defined in the camera's reference frame (and thus has the camera as origin).

The complete list of parameters used in the integration is as follows:

1. Black Hole Spin a
2. Camera Position $(r_{cam}, \theta_{cam}, \phi_{cam})$
3. Camera Speed β
4. Camera Direction of Motion $(B_{\hat{r}}, B_{\hat{\theta}}, B_{\hat{\phi}})$
5. Incoming Ray Direction (θ_{rd}, ϕ_{rd})
6. Ray Constants of Motion (b, q)
7. Ray Position (r, θ, ϕ)
8. Ray Canonical Momenta, or 4-momentum $p = (p_t, p_r, p_\theta, p_\phi)$
9. Ray Origin on Celestial Sky (θ_{cs}, ϕ_{cs})

2 Kerr Metric

The way a photon travels through space-time is determined by the gravitational field it is in. A metric is the solution to the Einstein field equations that describe this gravitational field. In our case, metric defines spacetime around a Kerr black hole. We use the metric written in Boyer-Lindquist coordinates,

$$\begin{aligned}
&\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + (\rho^2/\Delta)\mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 + \varpi^2(\mathrm{d}\phi - \omega \mathrm{d}t)^2, \\
&\quad \text{with } \alpha = \frac{\rho\sqrt{\Delta}}{\Sigma}, \quad \omega = \frac{2ar}{\Sigma^2}, \quad \varpi = \frac{\Sigma \sin\theta}{\rho}, \\
&\quad \Sigma = \sqrt{(r^2+a^2)^2 - a^2\Delta \sin^2\theta}, \\
&\quad \rho = \sqrt{r^2 + a^2\cos^2\theta}, \quad \Delta = r^2 - 2r + a^2.
\end{aligned} \tag{1}$$

This equation describes the line element ds, which can be thought of as an infinitesimal displacement vector in the metric. From this equation, it is not clear how to find the paths of light through space. The next section shows a different notation, in the form of differential equations, that enable us to compute in which direction a photon will travel, from any point in space.

3 Differential Equations

We know that photons, or light rays, travel along a special type of geodesics called null (light-like) geodesics, which provide the fastest way from position A to position B. This reduces the complexity of the metric, as we do not need to consider any other geodesics. The solution of Einstein field equations can instead be written as the following set of differential equations:

$$\begin{aligned}
&\frac{\mathrm{d}r}{\mathrm{d}\zeta} = \frac{\Delta}{\rho^2} p_r, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \frac{1}{\rho^2} p_\theta, \\
&\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = -\frac{\delta}{\delta b}\left(\frac{R+\Delta\Theta}{2\Delta\rho^2}\right), \\
&\frac{\mathrm{d}p_r}{\mathrm{d}\zeta} = \frac{\delta}{\delta r}\left[-\frac{\Delta}{2\rho^2}p_r^2 - \frac{1}{2\rho^2}p_\theta^2 + \left(\frac{R+\Delta\Theta}{2\Delta\rho^2}\right)\right], \\
&\frac{\mathrm{d}p_\theta}{\mathrm{d}\zeta} = \frac{\delta}{\delta \theta}\left[-\frac{\Delta}{2\rho^2}p_r^2 - \frac{1}{2\rho^2}p_\theta^2 + \left(\frac{R+\Delta\Theta}{2\Delta\rho^2}\right)\right],
\end{aligned} \tag{2}$$

which are the super-Hamiltonian variant of the null-geodesic equations, with ζ an affine parameter which can be thought of as distance along the ray. Θ, P and R are defined as:

$$\begin{aligned}
&\Theta = q - \cos^2\theta\left(\frac{b^2}{sin^2\theta} - a^2\right), \\
&P = r^2 + a^2 - ab, \quad R = P^2 - \Delta\left[(b-a)^2 + q\right].
\end{aligned} \tag{3}$$

b and q are the two constants of motion of the ray, they are needed for integration, as well as to define whether the ray comes from the direction of the black hole shadow. They are computed with the ray's starting position and its canonical momenta $(p_t, p_r, p_\theta, p_\phi)$. In the next section we show how to calculate all these parameters.

In our implementation we use Runge-Kutta-Cash-Karp [3] with adaptive stepsize to numerically integrate the differential equations.

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4 Momenta and Constants of Motion

The equations in this section are only calculated once per ray, before tracing, when the ray position is equal to the camera position. Consequently, every (r,θ,ϕ) can be replaced with $(r_{cam},\theta_{cam},\phi_{cam})$ in this section.

The ray direction (θ_{rd},ϕ_{rd}) of the ray is defined in the proper reference system of the camera. However, to compute the geodesic route correctly, we need to transform the ray's properties to the reference frame of a local non-rotating observer, a *fiducial observer* (FIDO). Only after this, we can compute the canonical momenta and constants of motion.

First we compute the unit vector N that points from the camera to the incoming ray that we are tracing, in Cartesian coordinates:

$$\begin{aligned} N_x &= \sin\theta_{rd}\cos\phi_{rd}, \\ N_y &= \sin\theta_{rd}\sin\phi_{rd}, \\ N_z &= \cos\theta_{rd}. \end{aligned} \tag{4}$$

This vector is then transformed to Cartesian coordinates in the reference frame of the FIDO. Because the speed of the camera is very high close to a black hole, relativistic aberration has to be taken into account.

$$\begin{aligned} n_{Fx} &= \frac{-\sqrt{1-\beta^2}N_x}{1-\beta Ny}, \quad n_{Fy} = \frac{-N_y+\beta}{1-\beta N_y}, \\ n_{Fz} &= \frac{-\sqrt{1-\beta^2}N_z}{1-\beta Ny} \end{aligned} \tag{5}$$

The FIDO coordinates are transformed to spherical coordinates:

$$\begin{aligned} n_{F\hat{r}} &= \frac{B_{\hat{\phi}}}{\kappa}n_{Fx} + B_{\hat{r}}n_{Fy} + \frac{B_{\hat{r}}B_{\hat{\theta}}}{\kappa}n_{Fz}, \\ n_{F\hat{\theta}} &= B_{\hat{\theta}}n_{Fy} - \kappa n_{Fz}, \\ n_{F\hat{\phi}} &= -\frac{B_{\hat{r}}}{\kappa}n_{Fx} + B_{\hat{\phi}}n_{Fy} + \frac{B_{\hat{\theta}}B_{\hat{\phi}}}{\kappa}n_{Fz}, \\ &\text{where } \kappa \equiv \sqrt{1-B_{\hat{\theta}}^2} = \sqrt{B_{\hat{r}}^2 + B_{\hat{\phi}}^2} \end{aligned} \tag{6}$$

And using these we compute the ray's canonical momenta:

$$\begin{aligned} p_t &= -1, \quad p_r = E_f\frac{\rho}{\sqrt{\Delta}}n_{F\hat{r}}, \\ p_\theta &= E_F\rho n_{F\hat{\theta}}, \quad p_\phi = E_F\varpi n_{F\hat{\phi}} \\ &\text{with} \quad E_F = \frac{1}{\alpha + \omega\varpi n_{F\hat{\phi}}}. \end{aligned} \tag{7}$$

Here $-p_t$ is the conserved energy of the ray, with the rest of the equations defined such that $-p_t$ can be set to unity. E_F is the energy from the ray as measured by the FIDO.

As can be seen in Eq. 2, the momenta p_θ and p_r change during integration, while the other two stay the same. p_ϕ stays constant due to the axial symmetry of the metric, which makes it one of the two constants of motion:

$$b = p_\phi, \quad q = p_\theta^2 + \cos^2\theta\left(\frac{b^2}{\sin^2\theta} - a^2\right) \tag{8}$$

5 Trapped Rays

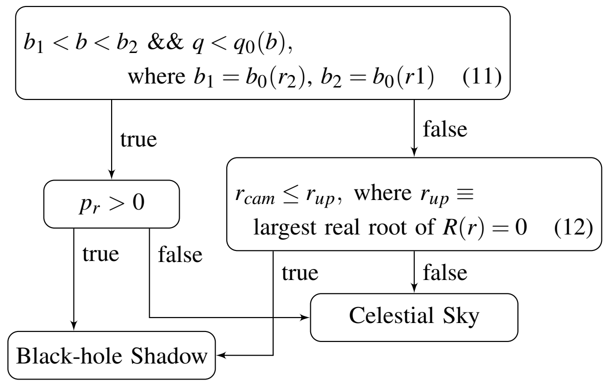
The only thing left to do before raytracing is to decide whether the ray comes from the celestial sky or from the black hole shadow region. For this we compare the ray's constants of motion with the functions that describe unstably trapped photons in orbits at a constant radius r. This function $q_0(b_0)$, is defined parametrically by

$$\begin{aligned} b_0 &\equiv -\frac{r_0^3 - 3r_0^2 + a^2r_0 + a^2}{a(r_0-1)}, \\ q_0 &\equiv -\frac{r_0^3\left(r_0^3 - 6r_0^2 + 9r_0 - 4a^2\right)}{a^2(r_0-1)^2} \end{aligned} \tag{9}$$

where r_0 is the radius of the trapped orbit, which runs over the interval $r_1 \le r_0 \le r_2$, with

$$\begin{aligned} r_1 &\equiv 2\left\{1+\cos\left[\frac{2}{3}\arccos(-a)\right]\right\}, \\ r_2 &\equiv 2\left\{1+\cos\left[\frac{2}{3}\arccos(+a)\right]\right\} \end{aligned} \tag{10}$$

With this information and the constants of motion of the ray, we decide based on the following flowchart, where every statement is tested for its truth.



The first part of Eq. 11 can be checked with Eqs. 9 and 10 immediately, but for the second one $q_0(b)$ needs to be computed from the parametrically-defined Eq. 9. This problem comes down to finding the root of the function $q_0(b_0 - b)$, or in other words, find the r_0 for which $q_0(b_0)$ and $q_0(b)$ equal. We chose to use the Newton-Raphson method, a quickly converging numerical root-finding algorithm, with a small adjustment to make sure the search stays on the right side of the asymptote of the function.

The check in Eq. 12 asks if $R(r)$ (eq. 3) crosses 0 at $r > r_{cam}$. As it can be verified that R is always > 0 at r_{cam} and goes to positive infinity as r increases, this is equal to checking whether the minimum of the function to the right side of r_{cam} is negative. We chose to implement a Golden-Section search [2] to find this minimum and then compare it with r_{cam}.

6 Redshift

The following formula determines the redshift (z) of the light frequency arriving via the integrated light ray.

$$1 + z = \left(\frac{1-\beta N_y}{\sqrt{1-\beta^2}}\right)\left(\frac{\alpha}{1-b\omega}\right) \tag{13}$$

It uses the α and ω defined in Eq. 1, the camera speed β, and of the incoming ray the constant of motion b (Eq. 8) and the y-direction N_y in camera reference frame (Eq. 4). The shift in color is caused by two things. The first is the frame dragging effect which propels photons around the black hole in the direction of its spin, giving them extra speed or slowing them down in the spin direction (ϕ). The second is the speed of the camera, how fast it moves towards or away from the light. Because of this, the effect depends firstly on the ϕ-momentum (the constant b is equal to the momentum in ϕ-direction at the camera position) and secondly on the component of the ray that is aligned with the camera motion N_y.

7 Camera in Equatorial Plane

The computations for the canonical momenta simplify significantly when we consider the special case, where the camera is rotating at a fixed distance in a stable orbit around the black hole. In this configuration, the camera direction is the unit vector in ϕ-direction, and the camera speed for a stable orbit can be calculated by:

$$B_{\hat{r}} = 0, \quad B_{\hat{\theta}} = 0 \quad B_{\hat{\phi}} = 1,$$
$$\beta = \frac{\varpi}{\alpha}(\Omega - \omega), \quad \text{where } \Omega = \frac{1}{\left(a + r_c^{3/2}\right)}. \tag{14}$$

In this special case the equations in Eq.6 simplify to:

$$N_{F\hat{r}} = n_{Fx}, \quad N_{F\hat{\theta}} = -n_{Fz}, \quad N_{F\hat{\phi}} = n_{Fy} \tag{15}$$

References

[1] O. James, E. von Tunzelmann, P. Franklin, and K. S. Thorne. Gravitational lensing by spinning black holes in astrophysics, and in the movie Interstellar. *Classical and Quantum Gravity*, 32(6):1–41, Feb. 2015. doi: 10.1088/0264-9381/32/6/065001

[2] J. Kiefer. Sequential minimax search for a maximum. *Proc.\ Amer. Math. Soc.*, 4:502–506, 1953. doi: 10.1090/S0002-9939-1953-0055639-3

[3] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press, 1988.