# Path tracing, part 2 of 2: Importance sampling

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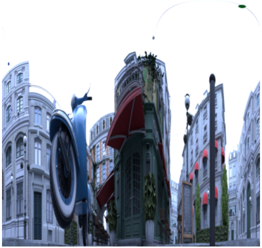
# The rendering equation: Integrand

$$L_o(\boldsymbol{x}, \omega_o) = L_e(\boldsymbol{x}, \omega_o) + \int_{\Omega(\boldsymbol{x})} L(\boldsymbol{x}, \omega_i)\, \rho(\boldsymbol{x}, \omega_i, \omega_o)\, \boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i \, \mathrm{d}\omega_i$$

Incoming radiance
$L(\boldsymbol{x}, \omega_i)$



BRDF
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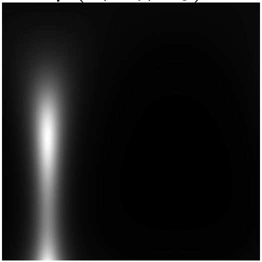

Geometry term
$\boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i$

# Path construction

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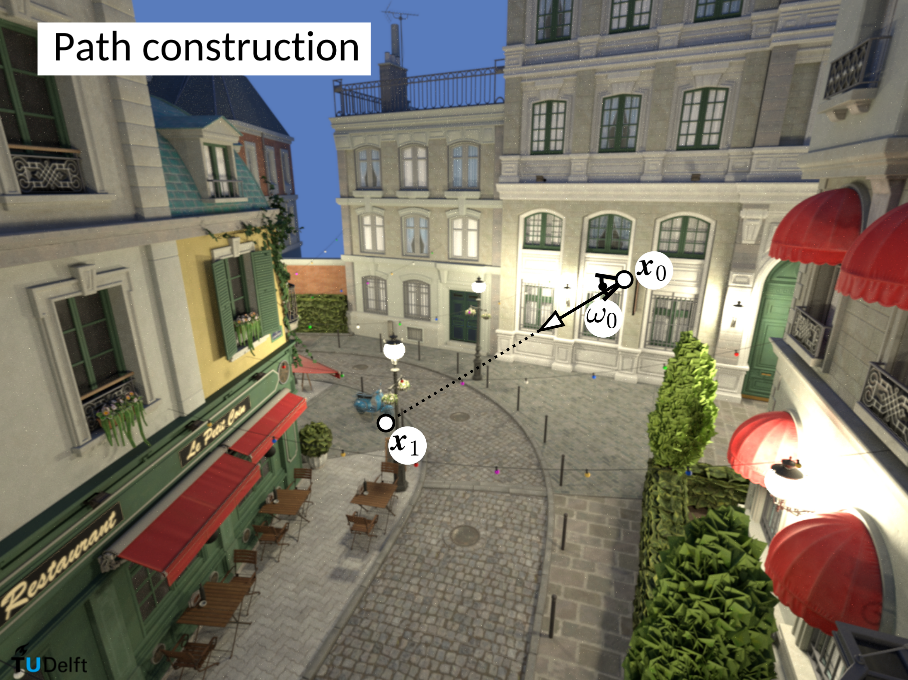


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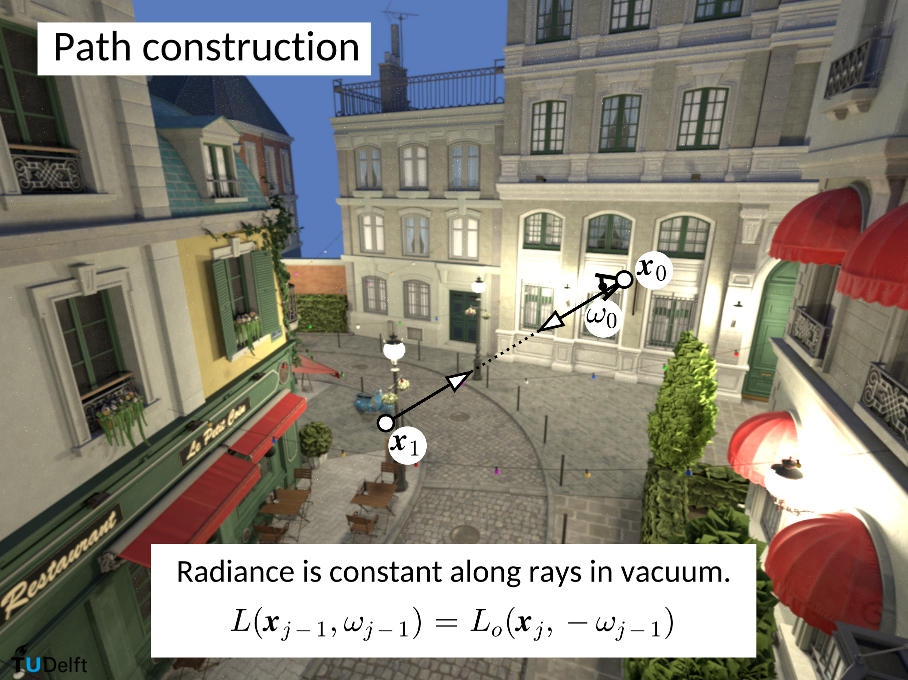


Radiance is constant along rays in vacuum.

$$L(\boldsymbol{x}_{j-1}, \omega_{j-1}) = L_o(\boldsymbol{x}_j, -\omega_{j-1})$$

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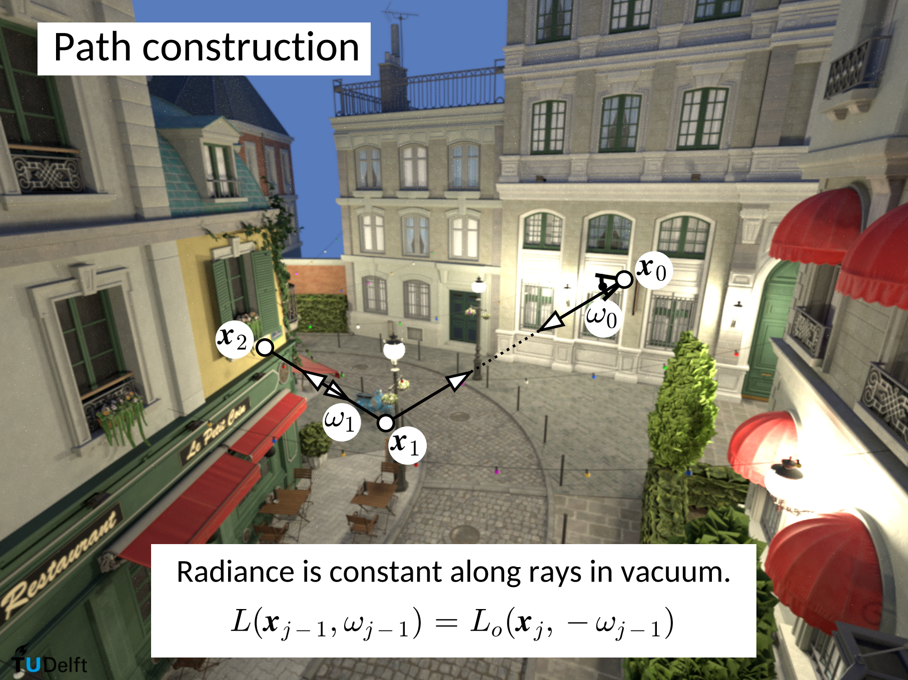


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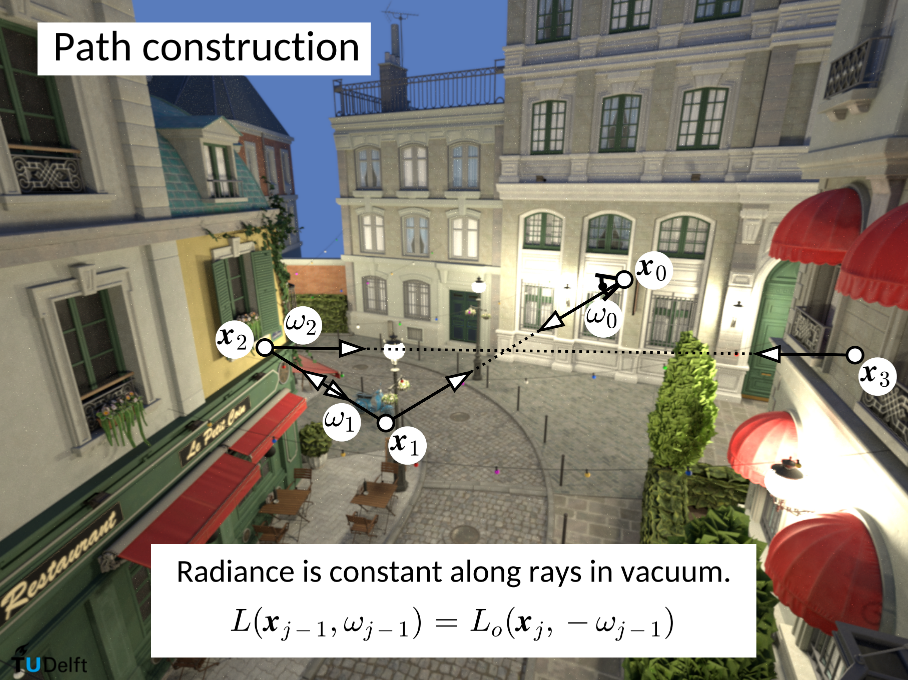


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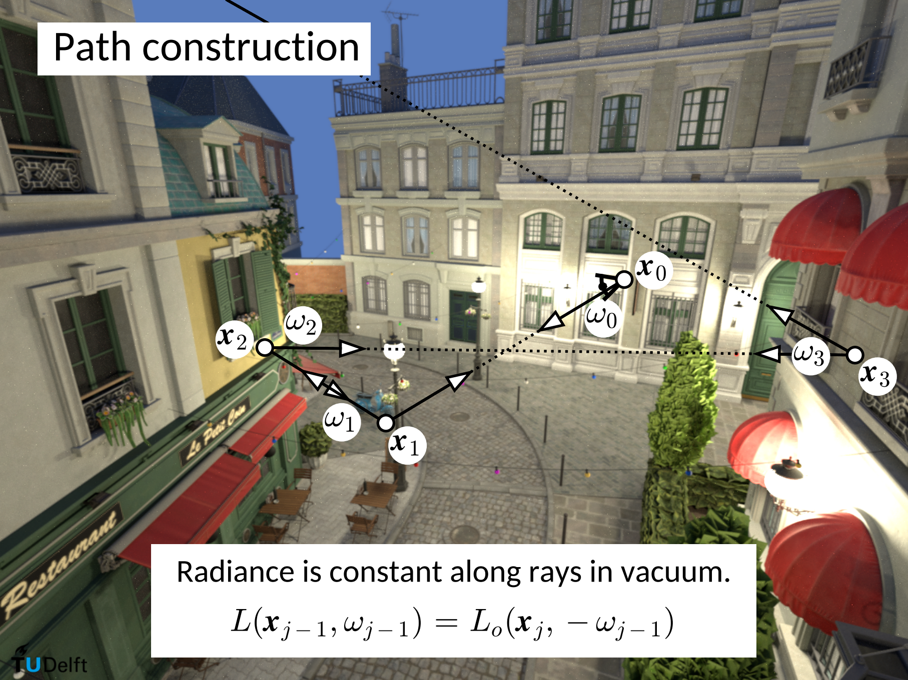


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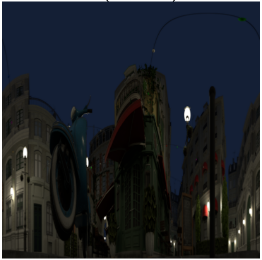

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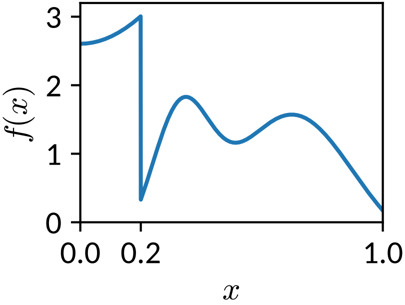
# Importance sampling

# Importance sampling: Idea

Idea: Sample regions with large values more frequently.

A rough guess is often good enough.



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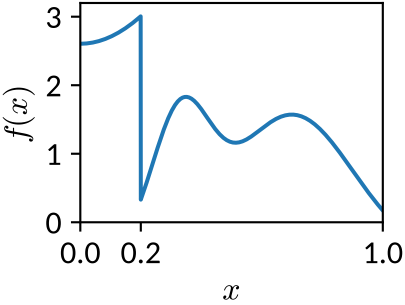

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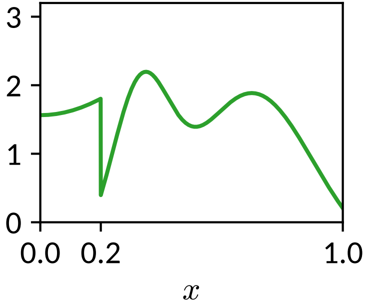
To compensate, samples with $x < 0.2$ get half the weight.

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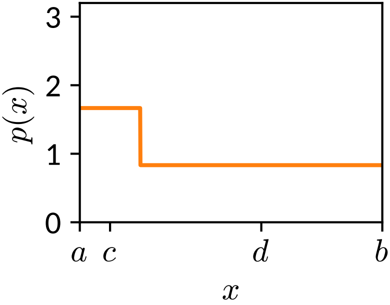
# Importance sampling: Definition

Use probability density function (PDF) $p(x) \geq 0$ with $x \in [a, b]$.

Take samples $x_0, \ldots, x_{N-1}$ with density $p(x)$.

Probability for sample $x_0$ to be in $[c, d]$: $\int_c^d p(x) \, \mathrm{d}x$



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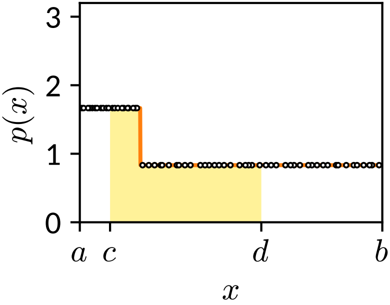

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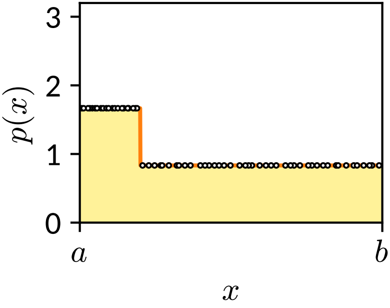

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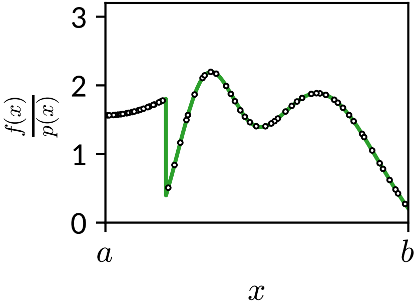

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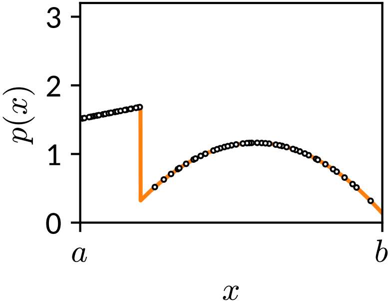

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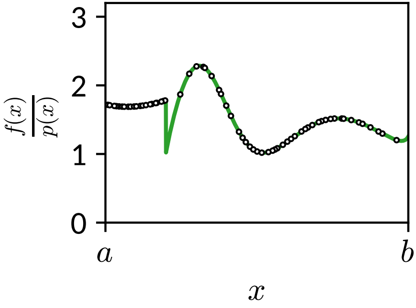
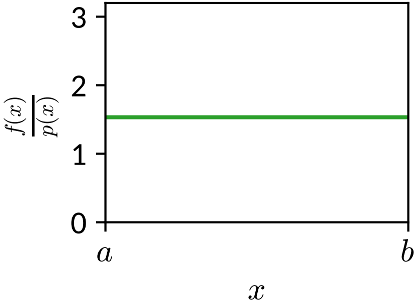


# Perfect importance sampling

Ideally $p(x) = wf(x)$ for a suitable normalization factor $w > 0$.

Then $\frac{f(x_j)}{p(x_j)} = \frac{f(x_j)}{wf(x_j)} = \frac{1}{w}$ is constant. No noise!



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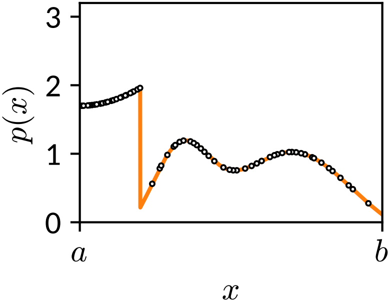

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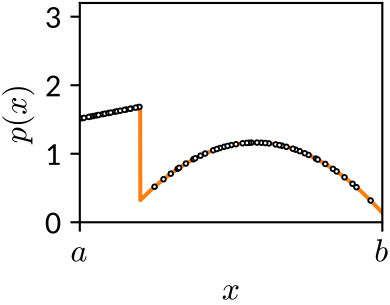
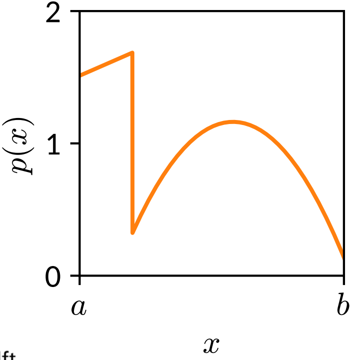
The density $p(x)$ must be a simplified version of $f(x)$.



# Inverse CDF sampling

Goal: Convert uniform $u$ in $[0, 1)$ to $x$ in $[a, b)$ with density $p(x)$.



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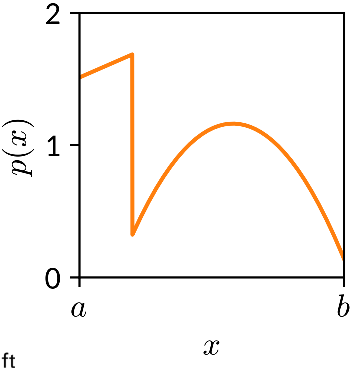


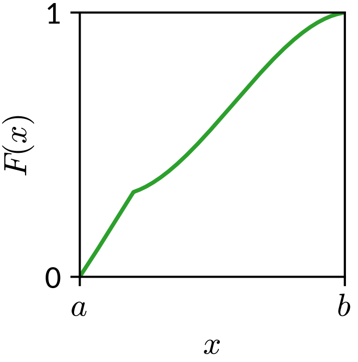

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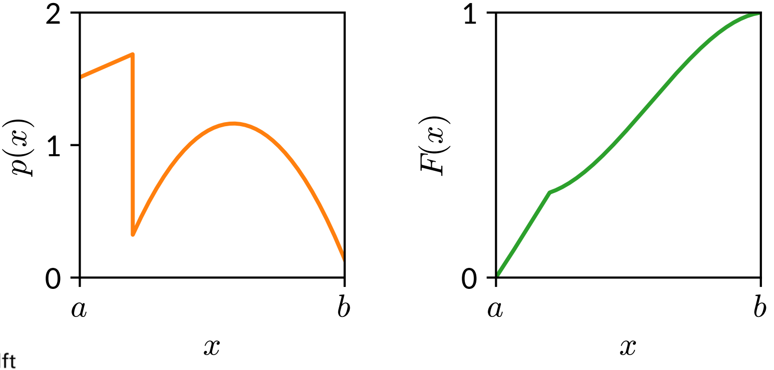

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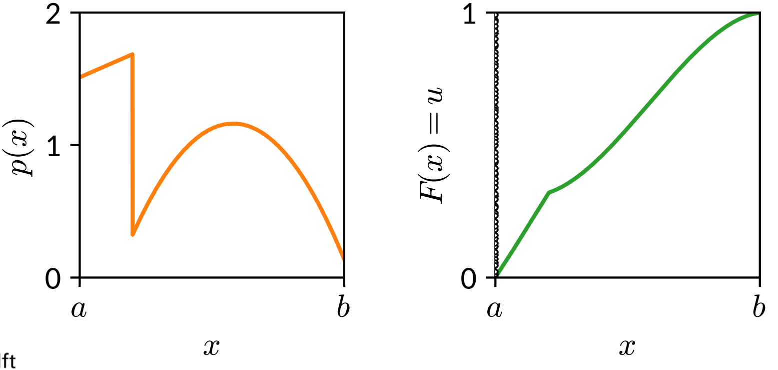

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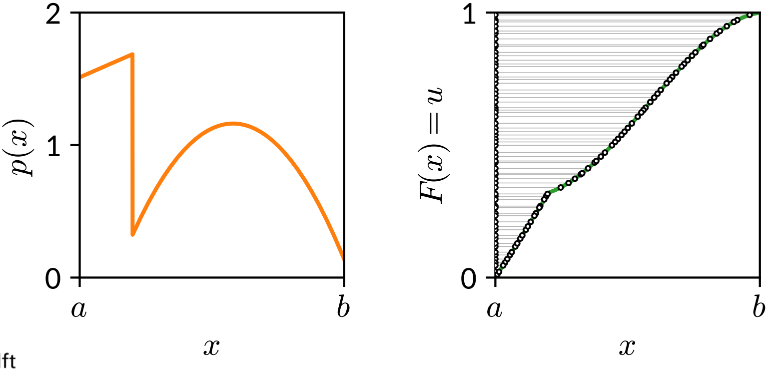

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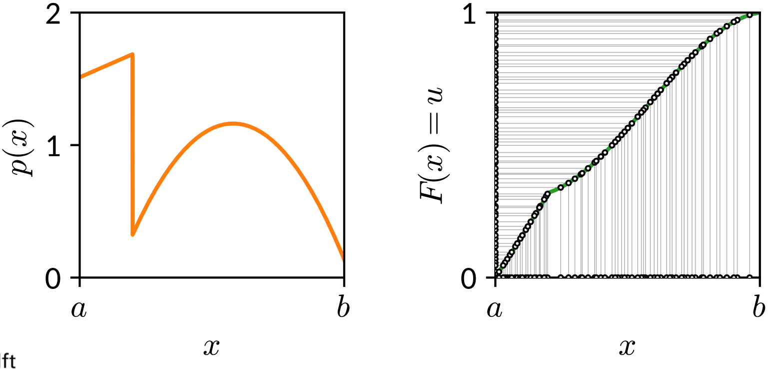

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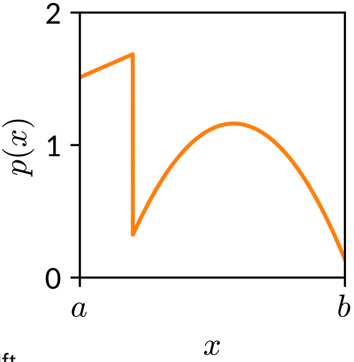


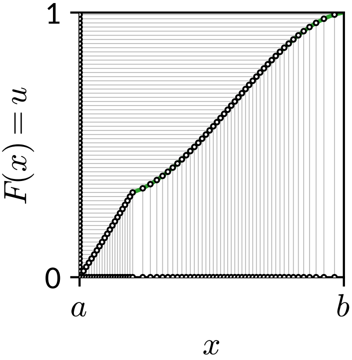


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# Correctness proof

Assume $p(x) > 0$ for simplicity.

Using the rule for derivatives of inverse functions:

$$(F^{-1})'(u) = \frac{1}{F'(F^{-1}(u))} = \frac{1}{p(F^{-1}(u))}$$

Using integration by substitution:

$$\int_0^1 \frac{f(F^{-1}(u))}{p(F^{-1}(u))}\,\mathrm{d}u = \int_0^1 f(F^{-1}(u))(F^{-1})'(u)\,\mathrm{d}u$$

$$= \int_{F^{-1}(0)}^{F^{-1}(1)} f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

# Importance sampling strategies

Which factor(s) should we use importance sampling for?

$$\int_{\Omega(\boldsymbol{x})} L(\boldsymbol{x}, \omega_i)\, \rho(\boldsymbol{x}, \omega_i, \omega_o)\, \boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i \, \mathrm{d}\omega_i$$

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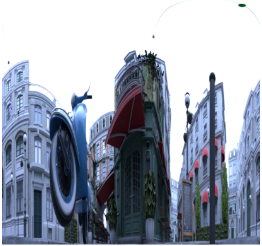

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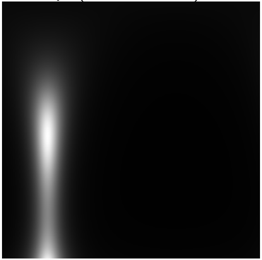

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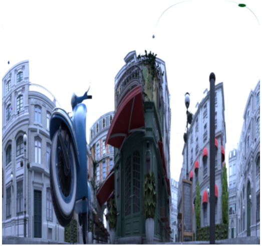

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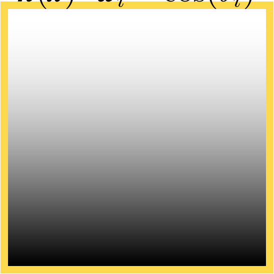


# BRDF importance sampling

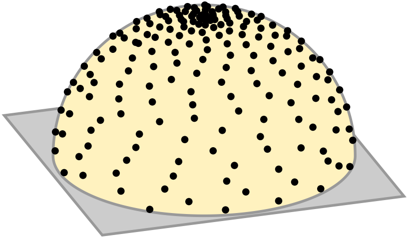
# Spherical vs. projected solid angle sampling

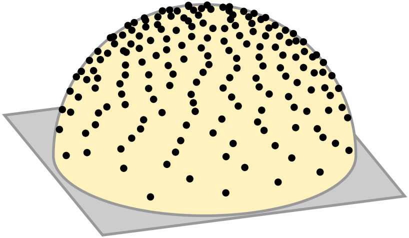
$$p(\theta) = \frac{2}{\pi}$$

$$p(\omega) = \frac{1}{\pi^2 \sin(\theta)}$$

$$p(\theta) = w \cos(\theta) \sin(\theta)$$

$$p(\omega) = w' \, \boldsymbol{n}(\boldsymbol{x}) \cdot \omega = w' \, \omega_{\mathrm{z}}$$





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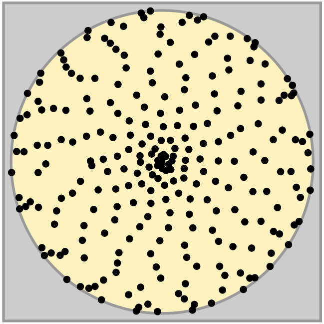

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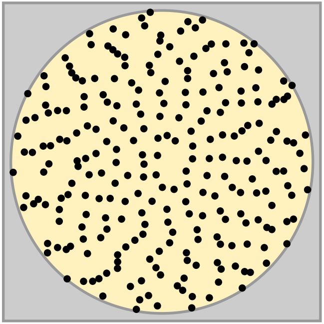

$(\omega_\mathrm{x}, \omega_\mathrm{y})$ is uniform.

Known as Nusselt analog.

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Solve: $u = F(\theta) \;\Leftrightarrow\; 1-u = \cos^2(\theta) \;\Leftrightarrow\; \theta = \arccos\sqrt{1-u}$

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$$= w[-\tfrac{1}{2}\cos^2(\theta)]_0^\Theta = \tfrac{w}{2}(1-\cos^2(\Theta)) = 1-\cos^2(\Theta)$$

Normalize: $F(\frac{\pi}{2}) = 1 \;\Rightarrow\; \frac{w}{2}(1-\cos^2(\frac{\pi}{2})) = 1 \;\Rightarrow\; w = 2$

Solve: $u = F(\theta) \;\Leftrightarrow\; 1-u = \cos^2(\theta) \;\Leftrightarrow\; \theta = \arccos\sqrt{1-u}$

Simplify: $\sin(\theta) = \sin\arccos\sqrt{1-u} = \sqrt{1-\sqrt{1-u}^2} = \sqrt{u}$

$$\cos(\theta) = \cos\arccos\sqrt{1-u} = \sqrt{1-u}$$

# Projected solid angle sampling

Goal: Sample $\omega \in \Omega$ proportional to $\omega_{\mathrm{z}}$.

$$\omega_{\mathrm{x}}(\theta, \varphi) := \cos(\varphi)\sin(\theta) = \cos(2\pi u_{\mathrm{x}} - \pi)\sqrt{u_{\mathrm{y}}}$$

$$\omega_{\mathrm{y}}(\theta, \varphi) := \sin(\varphi)\sin(\theta) = \sin(2\pi u_{\mathrm{x}} - \pi)\sqrt{u_{\mathrm{y}}}$$

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$$\omega_{\mathrm{z}}(\theta, \varphi) \coloneqq \cos(\theta) = \sqrt{1 - u_{\mathrm{y}}}$$

Density: $p(\omega) = w' \, \omega_{\mathrm{z}}$

Estimate: $\int_{\Omega} f(\omega) \, \mathrm{d}\omega \approx \frac{f(\omega(\boldsymbol{u}))}{p(\omega(\boldsymbol{u}))}$

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Density: $p(\omega) = w'\,\omega_\mathrm{z} = \frac{\omega_\mathrm{z}}{\pi}$ Estimate: $\int_\Omega f(\omega)\,\mathrm{d}\omega \approx \frac{f(\omega(\boldsymbol{u}))}{p(\omega(\boldsymbol{u}))}$

$$1 = \int_{-\pi}^{\pi}\int_0^{\frac{\pi}{2}} p(\omega(\theta,\varphi))\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi = \int_{-\pi}^{\pi}\frac{w'}{2}F(\tfrac{\pi}{2})\,\mathrm{d}\varphi = \pi w'$$

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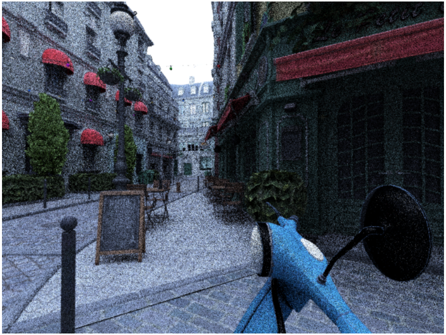
# Comparison

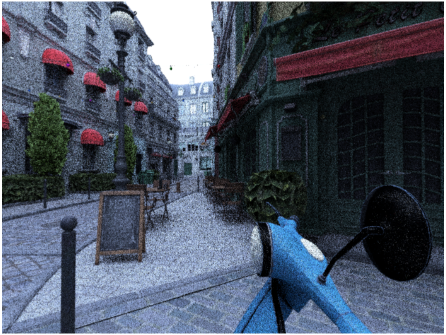
Both images use $N = 4$ samples per pixel.

Moderate but consistent noise reduction at no cost.

Uniform $\theta, \varphi$



Projected solid angle sampling

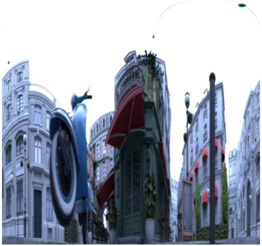


# Importance sampling strategies

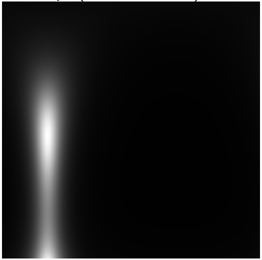
Which factor(s) should we use importance sampling for?

$$\int_{\Omega(\boldsymbol{x})} L(\boldsymbol{x}, \omega_i)\, \rho(\boldsymbol{x}, \omega_i, \omega_o)\, \boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i \, \mathrm{d}\omega_i$$

Incoming radiance
$L(\boldsymbol{x}, \omega_i)$



BRDF
$\rho(\boldsymbol{x}, \omega_i, \omega_o)$



Geometry term
$\boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i = \cos(\theta_i)$

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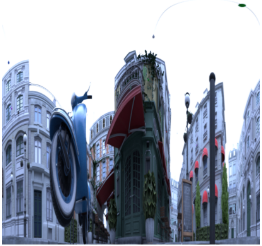

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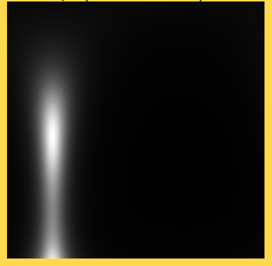

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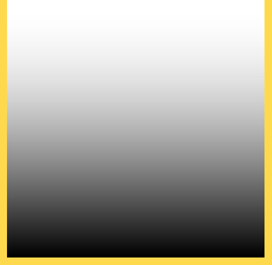


# Specular BRDF sampling

Slightly complicated but code is available.

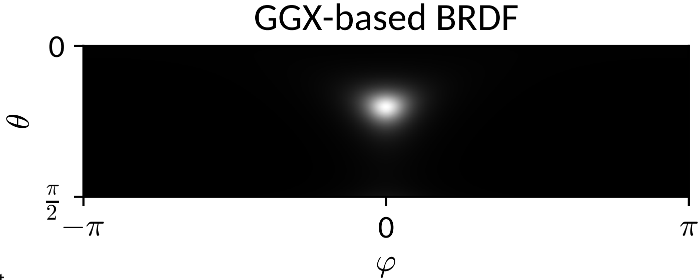
Method for microfacet BRDFs: doi.org/10.1111/cgf.12417

Faster method for GGX: doi.org/10.1145/2815618

Still faster method for GGX: doi.org/10.1111/cgf.14867

Roughness: 0.10

Outgoing inclination: 0.6



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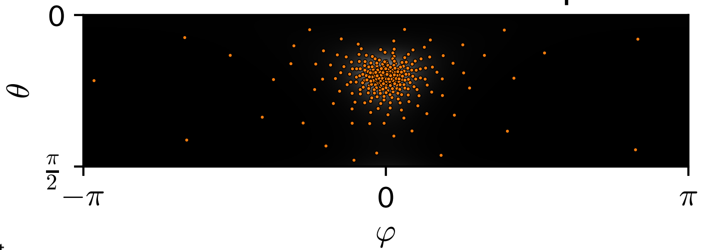
GGX-based BRDF with samples



# Combining sampling strategies

We have $m$ sampling strategies but only want one sample.

Strategy $j \in \{0, \ldots, m-1\}$ has density $p_j(x)$, importance $c_j$.

Total importance: $C := \sum_{j=0}^{m-1} c_j$

Goal: Pick strategy $j$ with probability $\frac{c_j}{C}$.

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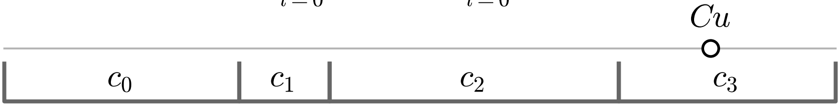

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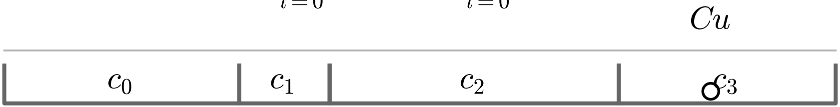

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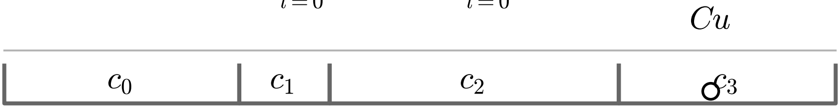



Combined density: $p(x) \coloneqq \frac{1}{C} \sum_{j=0}^{m-1} c_j p_j(x)$

# Combining diffuse and specular sampling

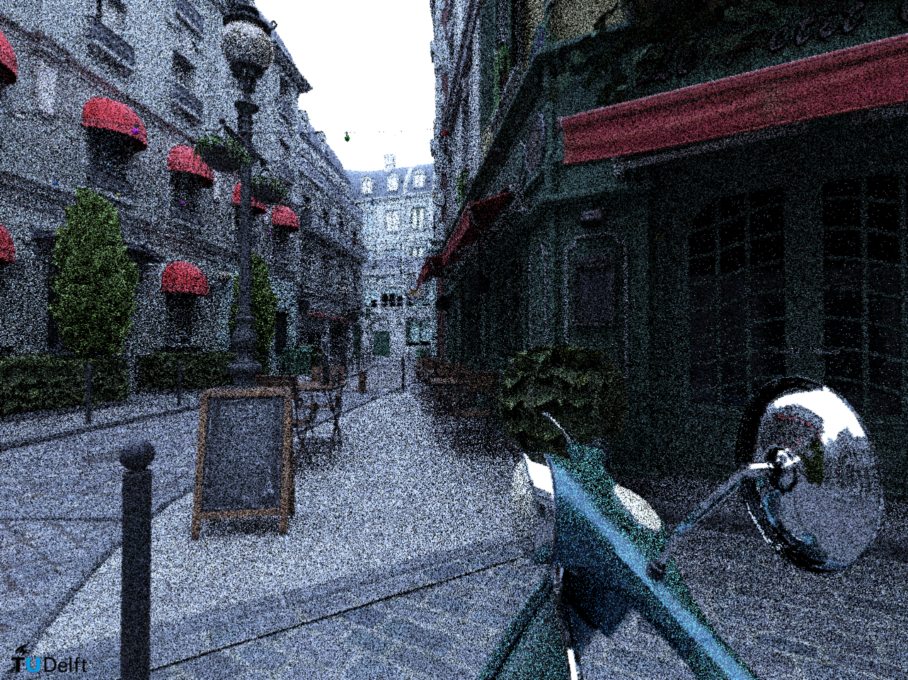
Probability for projected solid angle sampling:

The diffuse albedo, but at most 50%.

Otherwise specular BRDF sampling.

In specular highlights, specular is extremely important.

So we do not want to use its sampling strategy too seldomly.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# Light importance sampling

# Daytime, BRDF sampling, $N = 16$

# Nighttime, BRDF sampling, $N = 16$

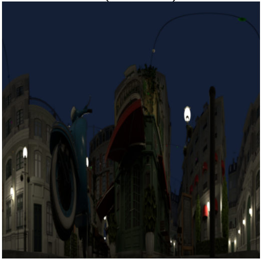
# Nighttime, BRDF+light sampling, $N = 16$

# Importance sampling strategies

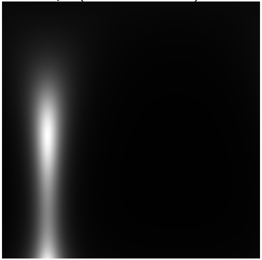
Which factor(s) should we use importance sampling for?

$$\int_{\Omega(\boldsymbol{x})} L(\boldsymbol{x}, \omega_i)\, \rho(\boldsymbol{x}, \omega_i, \omega_o)\, \boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i \, \mathrm{d}\omega_i$$

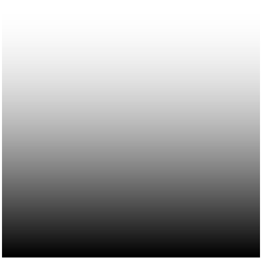
Incoming radiance
$L(\boldsymbol{x}, \omega_i)$



BRDF
$\rho(\boldsymbol{x}, \omega_i, \omega_o)$



Geometry term
$\boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i = \cos(\theta_i)$



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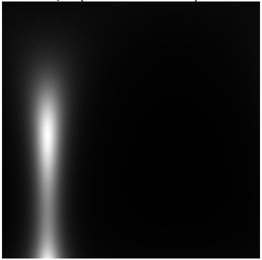


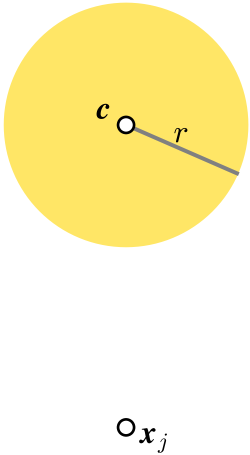
Geometry term
$\boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i = \cos(\theta_i)$

# Spherical lights

Light with radius $r$, center $\boldsymbol{c} \in \mathbb{R}^3$.

Choose a convenient coordinate frame:

Path vertex $\boldsymbol{x}_j$ is origin, $\boldsymbol{c}$ is on $\mathrm{z}$-axis.



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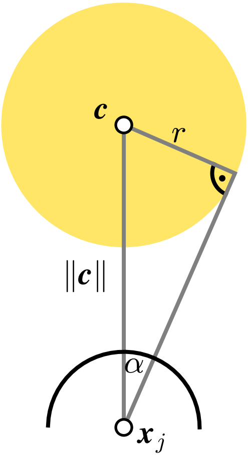
$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{r}{\|\boldsymbol{c}\|}$$

$$\alpha = \arcsin\left(\frac{r}{\|\boldsymbol{c}\|}\right)$$



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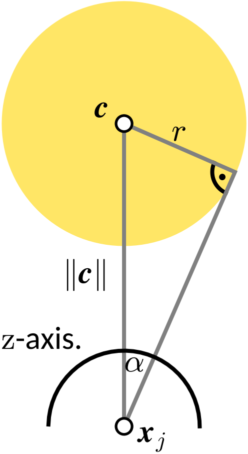


$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{r}{\|\boldsymbol{c}\|}$$

$$\alpha = \arcsin\left(\frac{r}{\|\boldsymbol{c}\|}\right)$$

Sample direction vectors within angle $\alpha$ of $z$-axis.

This is a so-called spherical cap.

# Sampling spherical caps

Goal: Sample $\theta \in [0, \alpha)$ with density $p(\theta) = w \sin(\theta)$.

CDF: $F(\Theta) = \int_0^\Theta p(\theta)\,\mathrm{d}\theta = w[-\cos(\theta)]_0^\Theta = w(1 - \cos(\Theta))$

Normalize: $1 = F(\alpha) = w(1 - \cos(\alpha)) \;\; \Rightarrow \;\; w = \frac{1}{1 - \cos(\alpha)}$

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Solve: $u = F(\theta) \;\Leftrightarrow\; (1 - \cos(\alpha))u = 1 - \cos(\theta)$

$$\Leftrightarrow\; \theta = \arccos(1 - (1 - \cos(\alpha))u)$$

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Solve: $u = F(\theta) \;\Leftrightarrow\; (1 - \cos(\alpha))u = 1 - \cos(\theta)$

$$\Leftrightarrow\; \theta = \arccos(1 - (1 - \cos(\alpha))u)$$

Simplify: $\cos(\alpha) = \cos(\arcsin(\frac{r}{\|\boldsymbol{c}\|})) = \sqrt{1 - \frac{r^2}{\|\boldsymbol{c}\|^2}}$

$$\omega_\mathrm{z} \coloneqq \cos(\theta) = 1 - (1 - \cos(\alpha))u$$

$$\sin(\theta) = \sqrt{1 - \omega_\mathrm{z}^2}$$

# Sampling spherical caps

Goal: Sample $\omega$ uniformly in spherical cap with angle $\alpha$.

$$z_{\min} \coloneqq \cos(\alpha) = \sqrt{1 - \frac{r^2}{\|\boldsymbol{c}\|^2}}$$

$$\omega_{\mathrm{z}} \coloneqq \cos(\theta) = 1 - (1 - z_{\min})\boldsymbol{u}_{\mathrm{y}}$$

$$\omega_{\mathrm{x}} \coloneqq \cos(\varphi)\sin(\theta) = \cos(2\pi\boldsymbol{u}_{\mathrm{x}} - \pi)\sqrt{1 - \omega_{\mathrm{z}}^2}$$

$$\omega_{\mathrm{y}} \coloneqq \sin(\varphi)\sin(\theta) = \sin(2\pi\boldsymbol{u}_{\mathrm{x}} - \pi)\sqrt{1 - \omega_{\mathrm{z}}^2}$$

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$$\omega_{\mathrm{y}} \coloneqq \sin(\varphi)\sin(\theta) = \sin(2\pi\boldsymbol{u}_{\mathrm{x}} - \pi)\sqrt{1 - \omega_{\mathrm{z}}^2}$$

Density: $p(\omega) = \frac{1}{2\pi(1 - z_{\min})}$

$2\pi(1 - z_{\min})$ is the solid angle of the spherical cap.

Solid angle $=$ area of shape projected onto unit sphere.

# Combining light sampling strategies

For $m$ lights, we get densities $p_0(\omega), \ldots, p_{m-1}(\omega)$.

Combine all these strategies as for diffuse and specular.

Importance $c_j :=$ solid angle times $L_e$ or zero if a light is outside the hemisphere $\Omega(\boldsymbol{x})$ (i.e. $\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{c} < -r$).

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Combine all these strategies as for diffuse and specular.

Importance $c_j :=$ solid angle times $L_e$ or zero if a light is outside the hemisphere $\Omega(\boldsymbol{x})$ (i.e. $\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{c} < -r$).

Incurs computations per light for each path vertex $\boldsymbol{x}_j$.

But the quality improvement justifies this cost (for ~30 lights).

Compute units are underutilized anyway.

The ray tracing cores are the bottleneck in this path tracer.

# Sampling other light shapes

Similar algorithms exist for lights of various shapes.

Solid angle sampling:

Polygons: doi.org/10.1145/218380.218500



Rectangles: doi.org/10.1111/cgf.12151

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Similar algorithms exist for lights of various shapes.

Solid angle sampling:

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Projected solid angle sampling:

Spheres: doi.org/10.1145/3320282



Polygons: doi.org/10.1145/3450626.3459672



Thin cylinders: doi.org/10.1111/cgf.14379



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# Multiple importance sampling (MIS)

# Combining BRDF and light sampling

For BRDF sampling, we have a strategy with density $p_b(\omega)$.

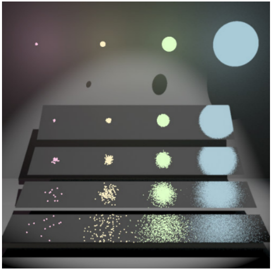
Fails for small lights in rough reflections.

For light sampling, we have a strategy with density $p_l(\omega)$.

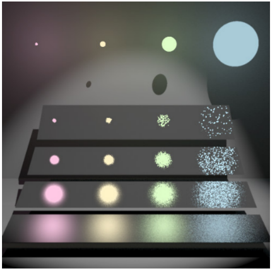
Fails for large lights in smooth reflections.

MIS works in all of these cases.

BRDF sampling



Light sampling



MIS



doi.org/10.1145/218380.218498

# MIS (with balance heuristic)

Take sample $\omega_b$ using the BRDF density $p_b(\omega)$.

Take sample $\omega_l$ using the light density $p_l(\omega)$.

MIS estimate: $\int_\Omega f(\omega)\,\mathrm{d}\omega \approx \frac{f(\omega_b)}{p_b(\omega_b)+p_l(\omega_b)} + \frac{f(\omega_l)}{p_b(\omega_l)+p_l(\omega_l)}$


doi.org/10.1145/218380.218498

# MIS (with balance heuristic)

Take sample $\omega_b$ using the BRDF density $p_b(\omega)$.

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Weights as if $\omega_b, \omega_l$ were sampled with a combined strategy.

Similar to the single-sample combination of strategies.



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Weights as if $\omega_b, \omega_l$ were sampled with a combined strategy.

Similar to the single-sample combination of strategies.

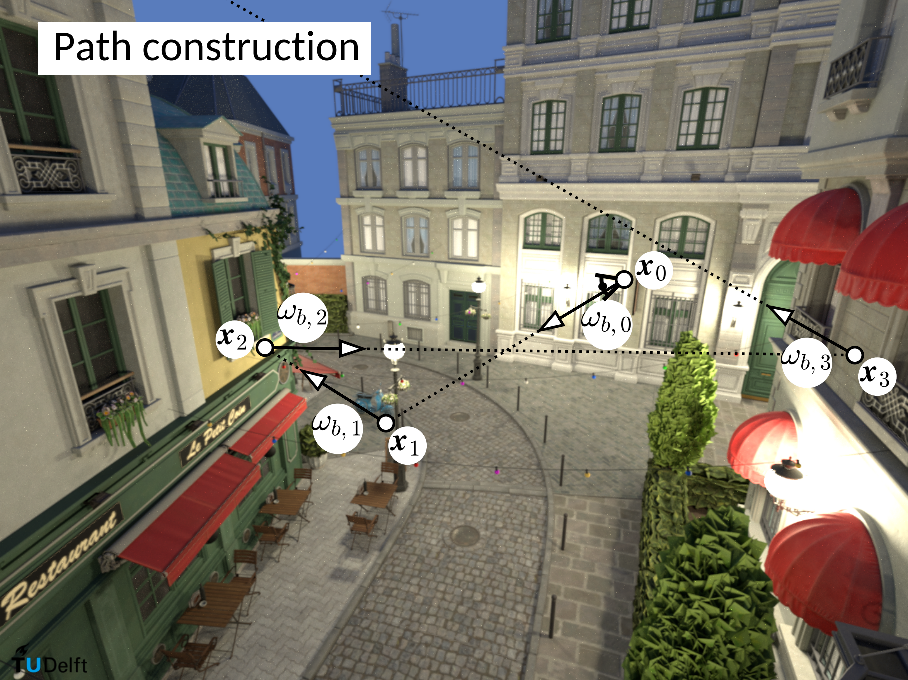
Density computation must work independently of sampling.

Otherwise we cannot compute $p_l(\omega_b)$ and $p_b(\omega_l)$.

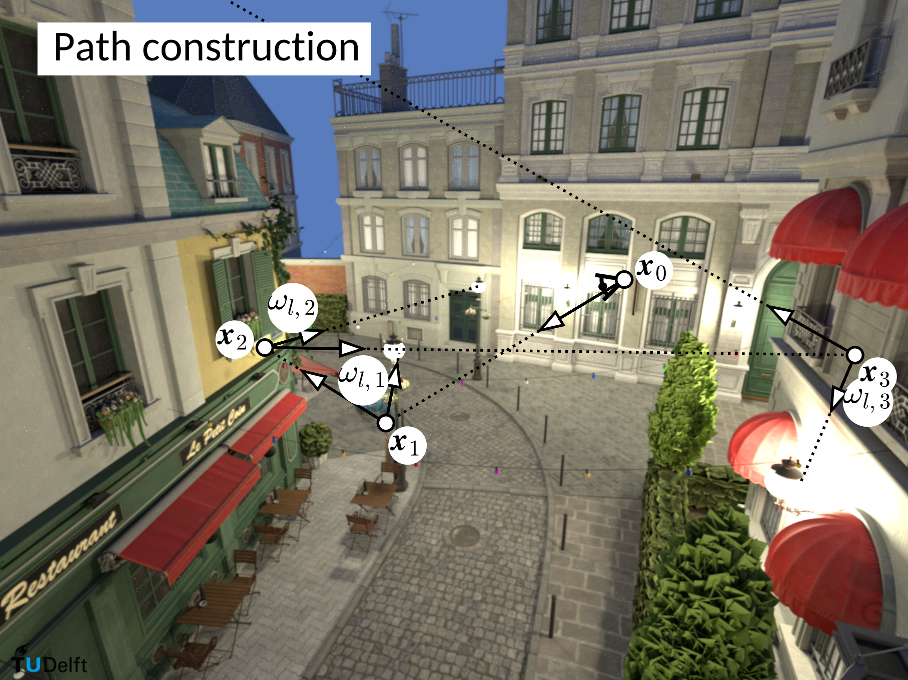

doi.org/10.1145/218380.218498

# Next event estimation (NEE)

# Path construction



# Path construction



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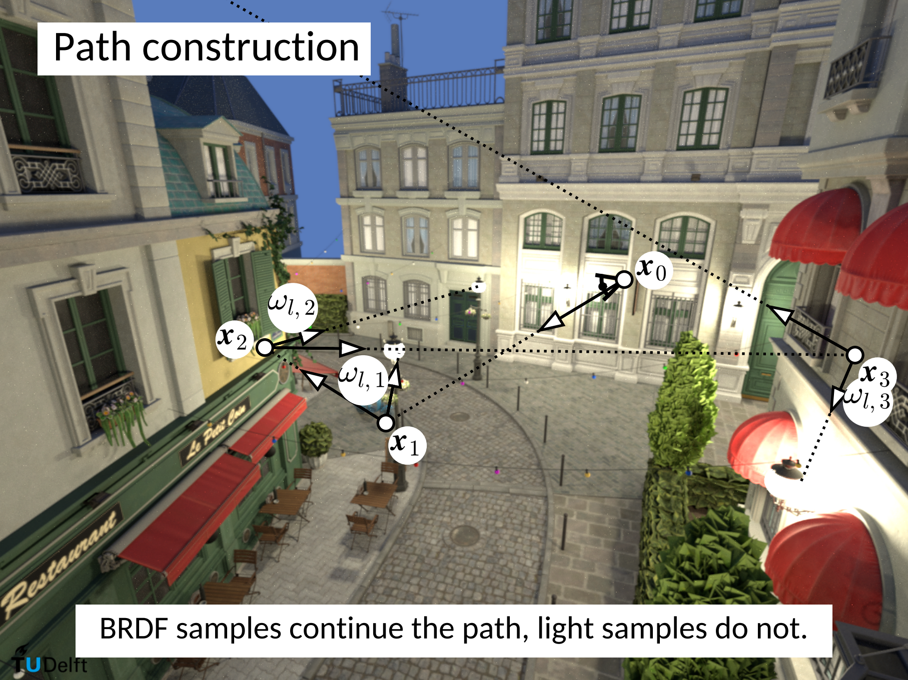


BRDF samples continue the path, light samples do not.

# Next event estimation (NEE)

Take apart the integrand into direct and indirect illumination.

Direct: $L_d(\boldsymbol{x}_k, \omega_k) \coloneqq L_e(\boldsymbol{x}_{k+1}, -\omega_k)$

Indirect/reflected: $L_r(\boldsymbol{x}_k, \omega_k) \coloneqq L(\boldsymbol{x}_k, \omega_k) - L_d(\boldsymbol{x}_k, \omega_k)$

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$$\int_{\Omega(\boldsymbol{x})} L(\boldsymbol{x}, \omega_i)\, \rho(\boldsymbol{x}, \omega_i, \omega_o)\, \boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i \, \mathrm{d}\omega_i$$

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Take apart the integrand into direct and indirect illumination.

Direct: $L_d(\boldsymbol{x}_k, \omega_k) \coloneqq L_e(\boldsymbol{x}_{k+1}, -\omega_k)$

Indirect/reflected: $L_r(\boldsymbol{x}_k, \omega_k) \coloneqq L(\boldsymbol{x}_k, \omega_k) - L_d(\boldsymbol{x}_k, \omega_k)$

$$\int_{\Omega(\boldsymbol{x})} \left(L_d(\boldsymbol{x}, \omega_i) + L_r(\boldsymbol{x}, \omega_i)\right) \rho(\boldsymbol{x}, \omega_i, \omega_o)\, \boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i \,\mathrm{d}\omega_i$$

# Next event estimation (NEE)

Take apart the integrand into direct and indirect illumination.

Direct: $L_d(\boldsymbol{x}_k, \omega_k) \coloneqq L_e(\boldsymbol{x}_{k+1}, -\omega_k)$

Indirect/reflected: $L_r(\boldsymbol{x}_k, \omega_k) \coloneqq L(\boldsymbol{x}_k, \omega_k) - L_d(\boldsymbol{x}_k, \omega_k)$

$$\int_{\Omega(\boldsymbol{x})} (L_d(\boldsymbol{x}, \omega_i) + L_r(\boldsymbol{x}, \omega_i))\, \rho(\boldsymbol{x}, \omega_i, \omega_o)\, \boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i \,\mathrm{d}\omega_i$$

$\approx$ MIS estimate for $\int_{\Omega(\boldsymbol{x})} L_d(\boldsymbol{x}, \omega_i)\, \rho(\boldsymbol{x}, \omega_i, \omega_o)\, \boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i \,\mathrm{d}\omega_i$

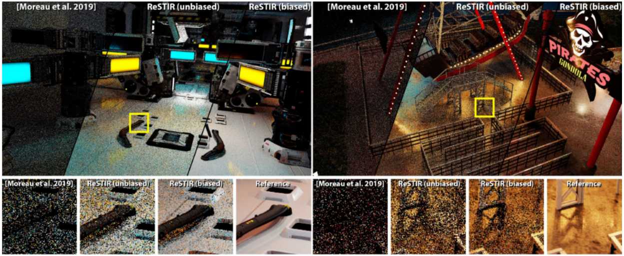
$+ \, p_b(\omega)$-estimate for $\int_{\Omega(\boldsymbol{x})} L_r(\boldsymbol{x}, \omega_i)\, \rho(\boldsymbol{x}, \omega_i, \omega_o)\, \boldsymbol{n}(\boldsymbol{x}) \cdot \omega_i \,\mathrm{d}\omega_i$

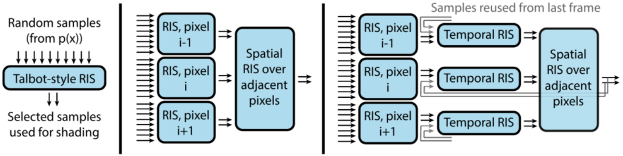
# Outlook

# Light sampling for many lights: ReSTIR

Learns a good sampling density by exploiting coherence.

Reuses samples from neighboring pixels and previous frames.






doi.org/10.1145/3386569.3392481

# Spatiotemporal variance-guided filtering (SVGF)

Denoising "blurs away" the noise.

Designed to preserve image features as much as possible.

Filtering across pixels and across frames.



doi.org/10.1145/3105762.3105770

# Stratification

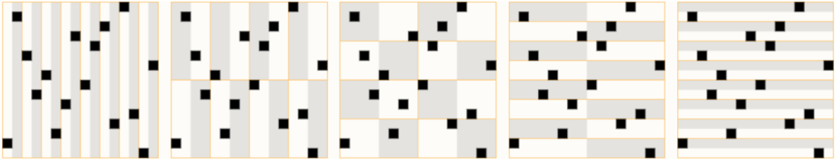
Random number generation is too important to be left to chance. —Robert Coveyou

Stratified sampling preserves benefits of randomization.

But certain subsets always get exactly one sample.

Simply use stratified "random" numbers instead of PRNG.

Ideally: Lower variance at same sample count.




doi.org/10.1145/3450626.3459880

# Conclusions

Path tracing is…

general: All light transport in one framework.

unbiased: Gives correct image + noise.

scalable: More samples, higher quality.

parallelizable: Across samples and pixels.

efficient: With good sampling strategies.

the default in offline rendering.

the future of real-time rendering.

# Conclusions

Real-time path tracing needs…

wider availability of good ray tracing hardware.

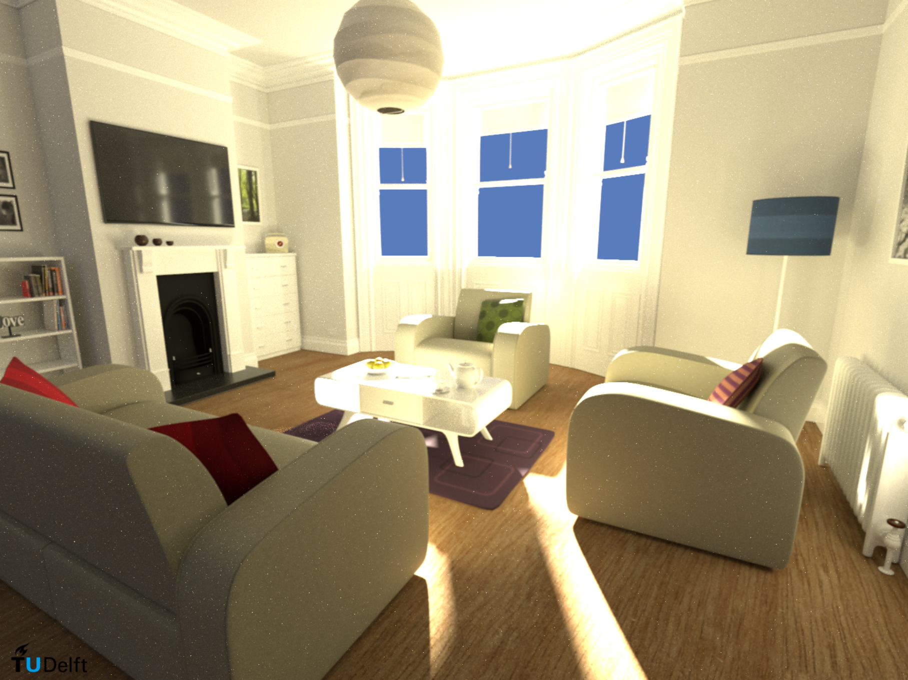
better importance sampling for indirect light.

faster builds for acceleration structures.

methods for level of detail.

more research…

better understanding among developers.


TUDelft