Supplementary Material

Spectral Processing of Tangential Vector Fields

The Hodge–Laplace Operator for Vector Fields

In this supplementary material, we derive a formula for the Hodge–Laplace operator for vector fields on surfaces in terms of the classic differential operators gradient, divergence, and curl and the rotation operator J (the almost complex structure). Using these operators the Hodge–Laplacian can be written as

$$\mathbf{\Delta} = -(\operatorname{grad}\operatorname{div} + \mathrm{J}\operatorname{grad}\operatorname{curl}). \tag{1}$$

We will play this formula back to the standard representation of the Hodge–Laplace operator for 1-forms. The representation in terms of grad, div and curl is useful for the discretization of the operator on spaces of vector fields (as an alternative the the discretization using 1-forms).

We denote the space of smooth k-forms on the surface $\mathcal{M}$ by Λ^k, the exterior derivative by $\mathrm{d}_k : \Lambda^k \mapsto \Lambda^{k+1}$ and the Hodge star operator by $*_k : \Lambda^k \mapsto \Lambda^{n-k}$. The Hodge–Laplace operator maps k-forms to k-forms. For an n-dimensional manifold, it is given by

$$\Delta_k = \mathrm{d}_{k-1}\delta_k + \delta_{k+1}\mathrm{d}_k,$$

where $\delta_k = (-1)^{n(k-1)+1} *_{n-k+1} \mathrm{d}_{n-k} *_k$. Hence, the general case can be written as

$$\Delta_k = (-1)^{n(k-1)+1}\mathrm{d}_{k-1} *_{n-k+1} \mathrm{d}_{n-k} *_k + (-1)^{n\,k+1} *_{n-k} \mathrm{d}_{n-k-1} *_{k+1} \mathrm{d}_k.$$

For the case of 1-forms on a surface, we get

$$\Delta_1 = -(\mathrm{d}_0 *_2 \mathrm{d}_1 *_1 + *_1 \mathrm{d}_0 *_2 \mathrm{d}_1). \tag{2}$$

The 0-forms are functions on the manifold, and, in the case of surfaces, all 2-forms can be represented as products of a function and the volume form. Using the Riemannian metric $\langle\cdot,\cdot\rangle$ on the surface, we can additionally get a one-to-one correspondence of vector fields and the 1-forms: to any vector field $\mathbf{v}$, we associate the 1-form $\langle\mathbf{v},\cdot\rangle$. With these identifications of functions and vector fields with the $0, 1$ and 2-forms on a surface, the operators on functions and vector fields can be expressed in terms of the exterior derivative and the Hodge star:

Fields	J	grad	curl	div
Forms	$*_1$	d_0	$*_2\mathrm{d}_1$	$*_2\mathrm{d}_1*_1$

By combining this table with (2) we obtain (1).