Generalized Shape Operators on Polyhedral Surfaces

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Abstract

This work concerns the approximation of the shape operator of smooth surfaces in $\mathbb{R}^3$ from polyhedral surfaces. We introduce two generalized shape operators that are vector-valued linear functionals on a Sobolev space of vector fields and can be rigorously defined on smooth and on polyhedral surfaces. We consider polyhedral surfaces that approximate smooth surfaces and prove two types of approximation estimates: one concerning the approximation of the generalized shape operators in the operator norm and one concerning the pointwise approximation of the (classic) shape operator, including mean and Gaussian curvature, principal curvatures, and principal curvature directions. The estimates are confirmed by numerical experiments.

1. Introduction

The approximation of curvatures of smooth surfaces from discrete surfaces plays an important role in various applications in geometry processing and related areas like physical simulation or computer graphics. Discrete curvatures on polyhedral surfaces have proved to work well in practice and convergence results in the sense of measures have been established, but estimates for pointwise approximation are still missing. Instead, pointwise estimates could only be established for special cases and negative answers and counterexamples to pointwise convergence for certain discrete curvatures have been reported, see Meek and Walton (2000); Borrelli et al. (2003); Xu (2004); Hildebrandt et al. (2006).

In this work, we introduce a generalization of the shape operator of smooth surfaces in $\mathbb{R}^3$ that can be defined for smooth and for polyhedral surfaces. The point of departure are two tensor fields on a smooth surface M:

$$\bar{S} : X \mapsto S(X^\top) - HN\,\langle X, N\rangle \quad \text{and} \quad \hat{S} : X \mapsto S(N \times X),$$

where S, H, and N denote the shape operator, the mean curvature, and the surface normal field of M and X is a vector field on M with tangential part $X^\top$. Both tensor fields have two properties: first, they have a simple weak form,

$$\int_M \bar{S}(X)\,\mathrm{d}vol = \int_M N \,\mathrm{div}\, X \,\mathrm{d}vol \quad \text{and} \quad \int_M \hat{S}\, X \,\mathrm{d}vol = -\int_M N \,\mathrm{curl}\, X \,\mathrm{d}vol,$$

and second, if at a point $x \in M$ the surface normal $N(x)$ and either $\bar{S}(x)$ or $\hat{S}(x)$ is known, then we can construct the shape operator $S(x)$ by simple algebraic operations.

The first property allows us to generalize the tensors to polyhedral surfaces. We introduce the generalized shape operators

$$\bar{\Sigma} : X \mapsto \int_M N \operatorname{div} X \, \mathrm{d}vol \quad \text{and} \quad \hat{\Sigma} : X \mapsto -\int_M N \operatorname{curl} X \, \mathrm{d}vol$$

that are vector-valued linear functionals on a Sobolev $H^{1,1}$-space of weakly differentiable vector fields and can be rigorously defined for smooth as well as for polyhedral surfaces.

To establish approximation estimates, we consider a polyhedral surface M_h that is close to a smooth surface M and use the orthogonal projection onto M to construct a bi-Lipschitz mapping between M_h and M. This map allows to pull-back objects from M_h to M, thus enables us to compare the objects. Our first approximation result shows that in the operator norm the approximation error of the generalized shape operators can be bounded by a constant times the spatial distance of the surfaces and the supremum of the difference of the surface normal vectors.

To get pointwise approximation estimates, we introduce the concept of r-local functions: for decreasing r, the support of such a function gets more and more localized around a point of the surface while the L^1-norm equals one and the $H^{1,1}$-norm grows proportionally to $\frac{1}{r}$. Based on r-local functions, we construct test vector fields for the generalized shape operators and use them to deduce estimates for the pointwise approximation of the tensors $\bar{S}$ and $\hat{S}$ from the estimates in the operator norm. By combining these estimates with approximation estimates for the surface normals, we obtain our main result: estimates for the pointwise approximation of the shape operator of a smooth surface from approximating polyhedral surfaces. The estimates are established in a general setting, *e. g.*, the vertices are not restricted to lie on the smooth surface, and are explicitly stated in terms of the spatial surface distance and the supremum of the difference of the surface normal vectors. Our approximation results are confirmed by a number of numerical experiments.

1.1. Related Work

The approximation of curvatures of smooth surfaces from discrete data is an active and exciting topic of research with a long history. Here, we can only briefly outline some of the work that has been most relevant for this paper. The curvature of a twice continuously differentiable surface in $\mathbb{R}^3$ is described by the shape operator. Together with the metric, the shape operator determines a smooth surface up to rigid motions. The metric on a polyhedral surface induced by ambient $\mathbb{R}^3$ is flat almost everywhere and has conical singularities at the vertices. The classical example of the lantern of Schwarz (1890) shows that if a polyhedral surface M_h is close to a smooth surface M in the Hausdorff distance, this does not imply that the area of M_h approximates the area of M.

Morvan and Thibert (2004) prove that for polyhedral surfaces that are inscribed to a smooth surface, the difference of the surface areas can be bounded by a constant times the maximum circumradius over all triangles of the polyhedral surface. Hildebrandt et al. (2006) show that if a sequence of polyhedral surfaces converges to a smooth surface in the Hausdorff distance, then the convergence of the normal vectors is equivalent to the convergence of the metrics of the surfaces. They extend this equivalence to the convergence of the Laplace–Beltrami operator in the corresponding operator norm. As a consequence, the mean curvature vector field converges in a weak (or integrated) sense. Approximation estimates for the Laplace–Beltrami operator for inscribed meshes were also obtained in a pioneering work by Dziuk (1988).

Since the normal field of a polyhedral surface in $\mathbb{R}^3$ is not differentiable, the classic notion of curvature cannot be applied to polyhedral surfaces. Cohen-Steiner and Morvan (2003) use the theory of normal cycles to define generalized curvatures for a broader class of surfaces that includes smooth and polyhedral surfaces. They prove that the generalized curvatures of a polyhedral surface, which is inscribed to a smooth surface, approximate the generalized curvatures of the smooth surface in the sense of measures. Pottmann et al. (2009) introduce integral invariants defined via distance functions as a robust way to approximate curvatures and present a stability analysis of integral invariants. In addition, they propose schemes for the efficient computation of the integral invariants. Generalized curvatures, integral invariants, and other discrete curvatures have been used for many applications in geometry processing, including: remeshing (Alliez et al. (2003); Kälberer et al. (2007); Bommes et al. (2009)), surface smoothing (Hildebrandt and Polthier (2004); Bian and Tong (2011)), registration (Gelfand et al. (2005)), surface matching (Huang et al. (2006)), and feature detection (Hildebrandt et al. (2005)).

Surface approximation with multivariate polynomials of order two or higher is an alternative approach for curvature estimation, see *e. g.* Meek and Walton (2000); Goldfeather and Interrante (2004); Cazals and Pouget (2005). Fitting polynomials to sample points of a smooth surface yields an approximation of the curvature at a point of the smooth surface, and the approximation order depends on the degree of the polynomial. However, multivariate polynomial fitting has two problems: one is that in addition to the polynomial degree used, the approximation error depends (in a complex manner) on the location of the samples; in an extreme case the polynomial to be fitted may not be unique, see Xu et al. (2005) for an example. Common practice to get an upper bound for this error is to restrict to certain types of sample point locations. The other problem is that since polynomial fitting relies on a high approximation order, the methods are sensitive against noise in the sampling data.

1.2. Contributions

The main contributions of the present work are: (i) we introduce generalized shape operators that are rigorously defined for smooth and for polyhedral surfaces, (ii) we establish error estimates for the approximation of the generalized shape operators of smooth surfaces from polyhedral surfaces in the operator norm, and (iii) we prove pointwise approximation estimates for the approximation of the (classic) shape operator (including Gaussian and mean curvature as well as principal curvatures and directions) of smooth surfaces from polyhedral surfaces. The estimates are derived in a general setting and they are explicitly stated in terms of the spatial distance of M and M_h and the supremum of the difference of the surface normal vectors.

The approximation results for the generalized curvatures, see Cohen-Steiner and Morvan (2003), can be compared to our approximation estimates for the generalized shape operators in the operator norm, where our setting is more general since we do not require inscribed polyhedral surfaces. Furthermore, we present pointwise approximation estimates which are still missing for the generalized curvatures.

Since we focus on polyhedral surfaces, our approach differs from multivariate polynomial fitting techniques, which use at least polynomials of degree two. In particular, our setting is based on a lower order of approximation than polynomial fitting approaches, but we still get pointwise approximation estimates. For example, if we consider polyhedral surfaces with mesh size h, then our estimates guarantee convergence of the shape operator if the normals of the polyhedral surface approximate the normals of the smooth surface with order $\mathcal{O}(h)$ (hence vertex positions with $\mathcal{O}(h^2)$), whereas polynomial fitting schemes require surface normals that converge with $\mathcal{O}(h^2)$ (hence vertex positions with $\mathcal{O}(h^3)$). This indicates that our discrete curvatures are more robust against noise in the vertex positions. Furthermore, our setting is more general since it does not restrict sample locations and does not require the points to lie on the surface. In addition, our approach can be combined with polynomial fitting techniques. For example, if an $\mathcal{O}(h^2)$ approximation of the surface normal field is known, this field could be used for the generalized shape operators instead of the piecewise constant normal field of the polyhedral surface and we would get the same approximation order as polynomial fitting schemes.

2. Analytic Preliminaries and Notation

In this work, M denotes a smooth and M_h a polyhedral surface in $\mathbb{R}^3$. Both surfaces are assumed to be compact, connected, and oriented. For statements that refer to both types of surfaces we denote the surface by $\mathcal{M}$.

Polyhedral surfaces. By a *polyhedral surface* in $\mathbb{R}^3$ we mean a finite set of planar triangles in $\mathbb{R}^3$ that are glued together in pairs along the edges such that the resulting shape is a two-dimensional manifold. The standard scalar product of $\mathbb{R}^3$ induces a metric g_{M_h} on a polyhedral surface M_h that is flat in the interior of all triangles and edges and has conical singularities at the vertices. More explicitly, every point of M_h has a neighborhood that is isometric to a neighborhood of the center point of a Euclidean cone with angle θ, which is the set

$$C_\theta = \{(r,\phi) \,|\, r \geq 0, \phi \in \mathbb{R}/\theta\mathbb{Z}\}/_{(0,\phi)\sim(0,\widetilde{\phi})}$$

equipped with the cone metric $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\phi^2$. For every vertex v of M_h the angle θ of the cone equals the sum of the angles at v of the triangles incident to v and for all other points the angle θ of the cone equals 2π. We define the distance between two points x and y in M_h as

$$d_{M_h}(x, y) = \inf_{\gamma} \mathrm{length}(\gamma),$$

where the infimum is taken over all rectifiable curves in $\mathbb{R}^3$ that connect x and y and whose trace is contained in M_h. With this distance M_h is a path metric space, and since M_h is compact, the Hopf–Rinow theorem for path metric spaces, see Gromov (1999), ensures that for any pair of points in M_h there exists a minimizing geodesic. We denote by $B_r(x)$ the open geodesic ball of radius r around the point x.

Function spaces. We denote by $L^p(\mathcal{M})$ and $H^{1,p}(\mathcal{M})$ the Lebesgue and Sobolev spaces on a smooth or polyhedral surface $\mathcal{M}$ and by $\| \ \|_{L^p}$, $\| \ \|_{H^{1,p}}$, and $| \ |_{H^{1,p}}$ the corresponding norms and semi-norms. For a definition and properties of Sobolev spaces on polyhedral surfaces we refer to Wardetzky (2006). By a vector field on $\mathcal{M}$, we mean a mapping from the surface to $\mathbb{R}^3$. We call a vector field C^∞, L^p, or $H^{1,p}$-regular if the three component functions are C^∞, L^p, or $H^{1,p}$-regular, and we denote the spaces of such vector fields by $\mathcal{X}(\mathcal{M})$, $\mathcal{X}_{L^p}(\mathcal{M})$ and $\mathcal{X}_{H^{1,p}}(\mathcal{M})$. The norms on the spaces $\mathcal{X}_{L^p}(\mathcal{M})$ and $\mathcal{X}_{H^{1,p}}(\mathcal{M})$ for $1 \leq p < \infty$ are:

$$\|X\|_{L^p}^p = \sum_{i=1}^{3} \|X_i\|_{L^p}^p \quad \text{and} \quad \|X\|_{H^{1,p}}^p = \sum_{i=1}^{3} \|X_i\|_{H^{1,p}}^p ,$$

where X_i are the components of X with respect to the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. On a smooth surface, we denote by $T_x M$ the tangent space and by $T_x^\perp M$ the normal space at a point $x \in M$. $T_x^\perp M$ is the one-dimensional subspace of $\mathbb{R}^3$ spanned by the surface normal vector $N(x)$ and $T_x M$ is the subspace of $\mathbb{R}^3$ consisting of all vectors that are orthogonal to $N(x)$ in $\mathbb{R}^3$. We denote by $\mathcal{X}^\top(M)$ and $\mathcal{X}^\perp(M)$ the subspaces of $\mathcal{X}(M)$ consisting of tangential and normal vector fields, and for a vector field $X \in \mathcal{X}(M)$, we denote by $X^\top$and $X^\perp$ its tangential and normal part. For a vector field $X \in \mathcal{X}_{L^1}(\mathcal{M})$ on a smooth or polyhedral surface $\mathcal{M}$, we define the vector-valued integral of X as

$$\int_{\mathcal{M}} X \,\mathrm{d}vol = \sum_{i=1}^{3} e_i \int_{\mathcal{M}} X_i \,\mathrm{d}vol.$$

Furthermore, for a symmetric tensor field A on M, we denote by $\|A\|_\infty$ the essential supremum of the function formed by the maximum of the absolute values of the eigenvalues of $A(x)$ over all $x \in M$.

Shape operator of smooth surfaces in $\mathbb{R}^3$. Let D denote the flat connection of $\mathbb{R}^3$. Since the surface normal field N of a smooth surface M has constant length and points in normal direction, for every $X \in \mathcal{X}^\top(M)$, the derivative $\mathrm{D}_X N$ is again a tangential vector field. The *shape operator* S is the tensor field that for any $x \in M$ is given by

$$S(x) : T_x M \mapsto T_x M \qquad (1)$$
$$v \mapsto -\mathrm{D}_v N.$$

A basic property of the shape operator is that for every $x \in M$, $S(x)$ is self-adjoint with respect to the scalar product on $T_x M$ inherited from $\mathbb{R}^3$. The eigenvalues and eigendirections of $S(x)$ are the principal curvatures $\kappa_i(x)$ and the principal curvature directions of M at x, and we call $\kappa_{\max}(x) = \max\{|\kappa_1(x)|, |\kappa_2(x)|\}$ the maximum curvature at x. Furthermore, the trace and the determinant of $S(x)$ are called the mean curvature and the Gaussian curvature of M at x and are denoted by $H(x)$ and $K(x)$. For our purposes, it is convenient to extend the shape operator to a tensor field on $M \times \mathbb{R}^3$ by setting

$$S(x)v = S(x)v^\top$$

for any $x \in M$ and $v \in \mathbb{R}^3$. We denote both tensor fields by S and rely on the context to make the distinction. The components of $S(x)$ with respect to the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ are given by

$$(S(x))_{ij} = \langle e_i, S(x)e_j \rangle_{\mathbb{R}^3}.$$

Divergence and curl of vector fields. For a weakly differentiable vector field X on a smooth or polyhedral surface $\mathcal{M}$, we define the *divergence* of X as

$$\operatorname{div} X = \sum_{i=1}^{3} \langle \operatorname{grad} X_i, e_i \rangle_{\mathbb{R}^3} \qquad (2)$$

and the *curl* of X as

$$\operatorname{curl} X = \sum_{i=1}^{3} \langle \operatorname{grad} X_i \times e_i, N \rangle_{\mathbb{R}^3}, \qquad (3)$$

where N is the normal field of the smooth or polyhedral surface and $\times$ is the cross product of $\mathbb{R}^3$. For tangential vector fields on smooth surfaces, this definition of divergence agrees with the usual divergence of 2-dimensional Riemannian manifolds, and the contribution of the normal component of a vector field X to the divergence has a simple geometric interpretation:

$$\operatorname{div} X = \operatorname{div} X^\top - H \langle X, N \rangle_{\mathbb{R}^3}. \qquad (4)$$

Furthermore, on a smooth surface the curl and the divergence of X are related by

$$\operatorname{curl} X = \operatorname{div}(X \times N), \qquad (5)$$

which implies that the curl of a normal vector field vanishes,

$$\operatorname{curl} X = \operatorname{curl} X^\top. \qquad (6)$$

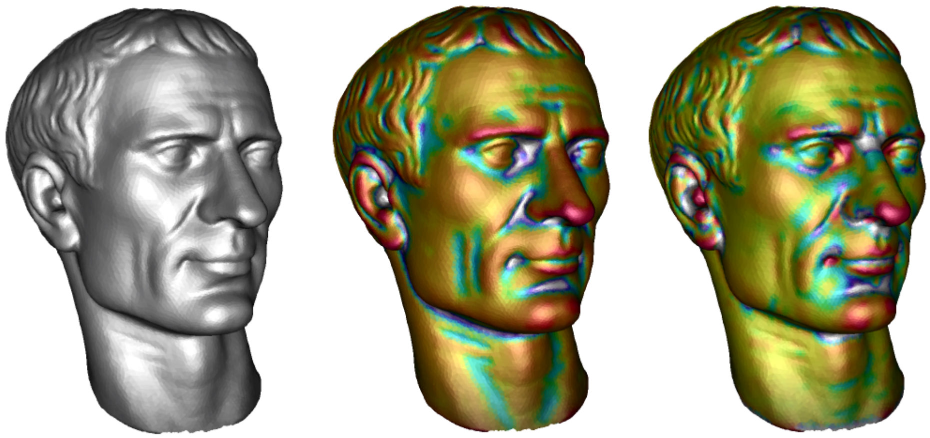

Figure 1: Mean curvature (middle) and Gaussian curvature (right) computed using the generalized shape operator $\hat{\Sigma}$ and an r-local function on a 3d-scanned model. Color coding from white (negative) to red (positive).

3. Generalized Shape Operators

In this section, we define two generalized shape operators for smooth and polyhedral surfaces, and we discuss the relation of these operators to the shape operator of smooth surfaces.

Definition 1. *We define the generalized shape operators $\bar{\Sigma}$ and $\hat{\Sigma}$ on a smooth or a polyhedral surface $\mathcal{M}$ to be the linear operators*

$$\bar{\Sigma} : \mathcal{X}_{H^{1,1}}(\mathcal{M}) \mapsto \mathbb{R}^3 \qquad X \mapsto \int_{\mathcal{M}} N \, div X \, dvol$$

and

$$\hat{\Sigma} : \mathcal{X}_{H^{1,1}}(\mathcal{M}) \mapsto \mathbb{R}^3 \qquad X \mapsto -\int_{\mathcal{M}} N \, curl X \, dvol.$$

The next lemma shows that $\bar{\Sigma}$ and $\hat{\Sigma}$ are elements of the normed space $L(\mathcal{X}_{H^{1,1}}(\mathcal{M}), \mathbb{R}^3)$ of continuous linear maps from $\mathcal{X}_{H^{1,1}}(\mathcal{M})$ to $\mathbb{R}^3$.

Lemma 2. *The operators $\bar{\Sigma}$ and $\hat{\Sigma}$ are continuous.*

Proof. Consider a vector field $X \in \mathcal{X}_{H^{1,1}}$. Using Hölder's inequality we have

$$\left\|\bar{\Sigma}(X)\right\|_{\mathbb{R}^3} = \left\| \int_{\mathcal{M}} N \, \mathrm{div} X \, \mathrm{d}vol \right\|_{\mathbb{R}^3} \leq \|N\|_{L^\infty} \|\mathrm{div}\, X\|_{L^1} \leq \|X\|_{H^{1,1}}$$

and

$$\left\|\hat{\Sigma}(X)\right\|_{\mathbb{R}^3} = \left\| \int_{\mathcal{M}} N \, \mathrm{curl}\, X \, \mathrm{d}vol \right\|_{\mathbb{R}^3} \leq \|N\|_{L^\infty} \|\mathrm{curl}\, X\|_{L^1} \leq \|X\|_{H^{1,1}}$$

which proves the lemma. ■

On a smooth surface, we consider two $(1,1)$-tensor fields on $M \times \mathbb{R}^3$:

$$\bar{S} : X \mapsto S(X^\top) - HN\,\langle X, N\rangle \tag{7}$$

and

$$\hat{S} : X \mapsto S(N \times X). \tag{8}$$

The tensors have the property that if at a point $x \in M$ the surface normal $N(x)$ and either $\bar{S}(x)$ or $\hat{S}(x)$ is known, one can construct the shape operator $S(x)$ by simple algebraic operations. The tensor $\bar{S}$ agrees with the shape operator S for tangential vector fields, and it multiplies the normal part of a vector field by the negative of the mean curvature. Applying the tensor field $\hat{S}$ to a vector field equals first removing the normal part, then rotating the remaining tangential vectors by $\frac{\pi}{2}$ in the corresponding tangent planes, and applying the shape operator S to the result. At a point $x \in M$, let b_1 and b_2 be unit vectors that point into principal curvature directions in T_xM. Then, in the basis $\{b_1, b_2, N\}$ of $\mathbb{R}^3$ the matrix representations of $\bar{S}(x)$ and $\hat{S}(x)$ are

$$\begin{pmatrix} \kappa_1(x) & 0 & 0 \\ 0 & \kappa_2(x) & 0 \\ 0 & 0 & -H(x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\kappa_2(x) & 0 \\ \kappa_1(x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The following lemma reveals the connection of the tensors $\bar{S}, \hat{S}$ and the operators $\bar{\Sigma}, \hat{\Sigma}$ and provides a justification of our definition of $\bar{\Sigma}$ and $\hat{\Sigma}$.

Lemma 3. *The tensor field $\bar{S}$ is the only $(1,1)$-tensor field on $M \times \mathbb{R}^3$ that satisfies*

$$\int_M \bar{S}\, X\; dvol = \bar{\Sigma}(X) \tag{9}$$

for all $X \in \mathcal{X}(M)$ and $\hat{S}$ is the only $(1,1)$-tensor field on $M \times \mathbb{R}^3$ that satisfies

$$\int_M \hat{S}\, X\; dvol = \hat{\Sigma}(X) \tag{10}$$

for all $X \in \mathcal{X}(M)$.

Proof. To show that the tensor field $\bar{S}$ fulfills equation (9), we apply the divergence theorem and use equation (4)

$$\begin{aligned} \int_M S\, X^\top \,\mathrm{d}vol = -\int_M \mathrm{D}_{X^\top} N \,\mathrm{d}vol &= \int_M N \operatorname{div} X^\top \,\mathrm{d}vol \\ &= \int_M N \operatorname{div} X \,\mathrm{d}vol + \int_M \langle HN, X\rangle_{\mathbb{R}^3} N \,\mathrm{d}vol. \end{aligned}$$

and to show that the tensor field $\hat{S}$ fulfills equation (10), we apply the divergence theorem and use equation (5)

$$\begin{aligned} \int_M \hat{S}\, X \,\mathrm{d}vol = \int_M S\,(N \times X) \,\mathrm{d}vol &= \int_M N \operatorname{div}(N \times X) \,\mathrm{d}vol \\ &= -\int_M N \operatorname{curl} X \,\mathrm{d}vol. \end{aligned}$$

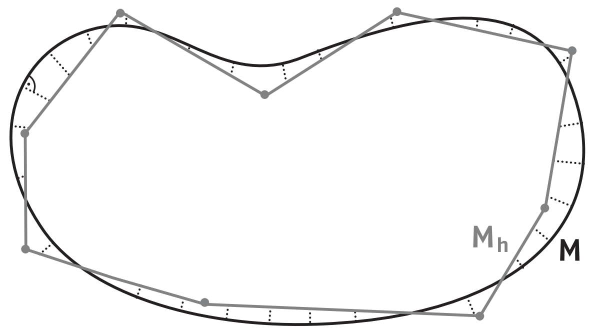

Figure 2: An illustration of the map between the smooth and the polyhedral surface is shown.

To proof uniqueness of the solution, let us assume that the tensor fields $\bar{S}$ and T are solutions of (9) for all $X \in \mathcal{X}(M)$. It follows that

$$\int_M (\bar{S} - T)\, X \,\mathrm{d}vol = 0$$

holds for all $X \in \mathcal{X}(M)$, which, by the fundamental lemma of calculus of variations, implies that $\bar{S}$ equals T. An analog argumentation shows the uniqueness of $\hat{S}$. ■

4. Approximation of Smooth Surfaces by Polyhedral Surfaces

In this section, we introduce the orthogonal projection onto a smooth surface M in $\mathbb{R}^3$ as a tool to construct a map between M and a polyhedral surface M_h nearby. We shall use this mapping to compare properties of the two surfaces and objects on them. The usage of this map is common practice, and similar results to those derived in this section can be found in Dziuk (1988), Morvan and Thibert (2004), and Hildebrandt et al. (2006).

Projection map. Let M be a compact smooth surface in $\mathbb{R}^3$. The distance function $\delta_M : \mathbb{R}^3 \mapsto \mathbb{R}_0^+$ is defined as

$$\delta_M(x) = \inf_{y \in M} \|x - y\|_{\mathbb{R}^3}\,. \tag{11}$$

Since M is compact, for every $x \in \mathbb{R}^3$ there is at least one point $y \in M$ that attains the minimum distance to x, *i. e.* $\delta_M(x) = \|x - y\|_{\mathbb{R}^3}$. Then, the straight line passing through x and y meets M orthogonally; therefore, y is called an *orthogonal projection* of x onto M. In general, y is not unique with this property; however, there exists an open neighborhood U_M of M in $\mathbb{R}^3$, such that every point of U_M has a unique orthogonal projection onto M. The induced *projection map* $\pi_M : U_M \mapsto M$ is smooth, a proof of this is contained in a note by Foote (1984).

Reach of a surface. The *reach* of M is the supremum of all positive numbers r such that the orthogonal projection onto M is unique in the open r-tube around M, where an open r-tube around M is the set of all points x in $\mathbb{R}^3$ that fulfill $\delta_M(x) < r$. Locally around a point $y \in M$, the reach equals the reciprocal of $\kappa_{\max}(y)$. As a consequence, the reach of M is bounded above by

$$\operatorname{reach}(M) \leq \inf_{y \in M} \frac{1}{\kappa_{\max}(y)}. \tag{12}$$

The inequality is strict, *e. g.* equality holds if M is a sphere in $\mathbb{R}^3$, but in general the reach additionally depends on global properties of the surface. Still, every embedded smooth surface has positive reach. For a general treatment of sets with positive reach we refer to the book of Federer (1969).

Differential of the projection map. Let us compute the differential of the projection map π_M. We consider an open neighborhood U_M of M that is a subset of the reach(M)-tube around M. First, we look at the signed distance function $\sigma_M : U_M \mapsto \mathbb{R}$ that is given by

$$\sigma_M(x) = \langle x - \pi_M(x), N(\pi_M(x)) \rangle_{\mathbb{R}^3}. \tag{13}$$

Differentiating this equation at a point x into a direction $v \in \mathbb{R}^3$ yields

$$\mathrm{d}_x \sigma_M(v) = \langle N(\pi_M(x)), v \rangle_{\mathbb{R}^3}, \tag{14}$$

where we apply the fact that the images of $\mathrm{D}N$ and $\mathrm{d}\pi_M$ are orthogonal to N and $x - \pi_M(x)$ is parallel to N. Using the signed distance function, we can represent the projection π_M by

$$\pi_M(x) = x - \sigma_M(x)\; N(\pi_M(x)). \tag{15}$$

Differentiation of this equation yields

$$\mathrm{d}\pi_M = Id - \mathrm{d}\sigma_M\, N \circ \pi_M - \sigma_M\, \mathrm{D}N \circ \mathrm{d}\pi_M.$$

Using the definition of the shape operator (1) and equation (14), we get

$$(Id - \sigma_M\, S)\mathrm{d}\pi_M = Id - \langle N \circ \pi_M, \cdot \rangle_{\mathbb{R}^3} N \circ \pi_M.$$

The right-hand side of this equation describes the orthogonal projection in $\mathbb{R}^3$ onto the tangent plane of M; we denote this map by $\bar{P}$. As a consequence of equation (12), the linear map $Id - \sigma_M(x)\, S_{\pi(x)}$ is bijective on $T_{\pi(x)}M$ for all $x \in U_M$. Thus, the inverse $(Id - \sigma_M(x)\, S_{\pi(x)})^{-1}$ exists and we have

$$\mathrm{d}\pi_M = (Id - \sigma_M\, S)^{-1} \bar{P}. \tag{16}$$

Furthermore, we can represent $(Id - \sigma_M\, S)^{-1}$ as

$$(Id - \sigma_M\, S)^{-1} = Id + R, \tag{17}$$

where the map R assigns to every point $x \in U_M$ the linear map on $T_{\pi(x)}M$ that is given by

$$R(x) = \sigma_M(x)\, S(\pi(x))\, (Id - \sigma_M(x)\, S(\pi(x)))^{-1}. \tag{18}$$

Normal graphs. Let us consider a polyhedral surface M_h that approximates a smooth surface M and use the orthogonal projection onto M to construct a map between the surfaces.

Definition 4. *A polyhedral surface M_h is a* normal graph *over a smooth surface M if M_h is a subset of the open reach(M)-tube around M and the restriction of the projection map π_M to M_h is a bijection. We denote the restricted projection map by Ψ.*

The following lemma lists some properties of Ψ.

Lemma 5. *The map Ψ is a homeomorphism of M_h and M, and for every triangle $T \in M_h$ the restriction of Ψ to the interior of T is a diffeomorphism onto its image.*

Proof. We first show that Ψ is a homeomorphism. Ψ is continuous, because it is the restriction to M_h of the smooth map π_M, and Ψ is bijective by assumption. It remains to show that Ψ is a closed map. Since M_h is compact, a closed subset A of M_h is compact; and since Ψ is continuous, $\Psi(A)$ is compact in M and hence $\Psi(A)$ is closed.

To prove the second part of the lemma, consider a triangle T of M_h and a point x in the interior of T. The differential of Ψ at x equals the restriction of $\mathrm{d}_x\pi_M$ to the tangent plane of T, from equation (16) we get

$$\mathrm{d}_x\Psi = (Id - \sigma_M(x)\ S(y))^{-1}P_x, \tag{19}$$

where P_x is the restriction of $\bar{P}$ to the tangent plane of T and $y = \Psi(x)$. The map P_x has full rank because by construction the tangent planes of T at x and of M at $\Psi(x)$ do not meet orthogonally and $Id - \sigma_M\, S$ has full rank by equation (12). This means that $\mathrm{d}_x\Psi$ has full rank and consequently the restriction of Ψ to the interior of T is a diffeomorphism onto its image. ■

Metric distortion. Let Φ denote the inverse map of Ψ, then Φ parametrizes M_h over M. We can use Φ to pull-back the cone metric of M_h to M. More precisely, we define a metric g_h in the pre-image of the union of the interior of all triangles of M_h, hence almost everywhere on M, by

$$g_h(X,Y) = \langle \mathrm{d}\Phi(X), \mathrm{d}\Phi(Y)\rangle_{\mathbb{R}^3} \qquad a.\,e., \tag{20}$$

where X and Y are tangential vector fields on M.

Let $y \in M$ be a point such that $x = \Phi(y)$ is in the interior of a triangle of M_h. Then, the tangent plane T_xM_h of M_h at x is well-defined and agrees with the plane of the triangle of M_h that contains x. The differential of Φ at y is a linear map $\mathrm{d}_y\Phi : T_yM \mapsto T_xM_h$, and $\mathrm{d}_x\Psi$ is its inverse. The adjoint of $\mathrm{d}_y\Phi$ is the linear map $\mathrm{d}_y\Phi^\times : T_xM_h \mapsto T_yM$ that fulfills

$$\langle \mathrm{d}_y\Phi(v), w\rangle_{\mathbb{R}^3} = \left\langle v, \mathrm{d}_y\Phi^\times(w)\right\rangle_{\mathbb{R}^3}$$

for all $v \in T_yM$ and $w \in T_xM_h$, and $\mathrm{d}_x\Psi^\times$, the adjoint of $\mathrm{d}_x\Psi$, is the inverse of $\mathrm{d}_y\Phi^\times$. All four linear maps are defined for almost all $y \in M$ (resp. $x \in M_h$). From equation (19) we get

$$\mathrm{d}_x\Psi^\times = P_x^\times(Id - \sigma_M(x)S(y))^{-1} \qquad a.\,e., \tag{21}$$

where $P_x^{\times}$, the adjoint of P_x, projects every vector $v \in T_y M$ orthogonally in $\mathbb{R}^3$ onto the tangent plane $T_x M_h$.

Let A denote the composition $\mathrm{d}\Phi^{\times} \circ \mathrm{d}\Phi$. Then, A satisfies

$$g_h(X, Y) = g\,(A\,X, Y) \qquad a.\,e., \tag{22}$$

which follows from eq. (20). We call A the *metric distortion tensor*. This tensor has been studied by Hildebrandt et al. (2006) and we refer to this work for additional properties of A. The volume form $\mathrm{d}vol_h$ induced by the metric g_h satisfies

$$\mathrm{d}vol_h = \alpha_h\,\mathrm{d}vol \qquad a.\,e., \tag{23}$$

where $\alpha_h = \sqrt{\det(A)}$. Explicitly, α_h is given by

$$\alpha_h(y) = \frac{1 - \sigma_M(x)\,\frac{1}{2}H(y) + \sigma_M^2\,(x)K(y)}{\langle N(y), N_{M_h}(x)\rangle} \qquad a.\,e., \tag{24}$$

which follows from the representation of $\mathrm{d}\Psi$ and $\mathrm{d}\Psi^{\times}$ given in equations $\tilde{(}19)$ and (21).

Function spaces. The map Ψ can be used to pull-back any function u on M to a function $u \circ \Psi$ on M_h. Wardetzky (2006) showed that this pull-back of functions induces an isomorphism of the L^p- and $H^{1,p}$-spaces on M and M_τ for $1 \leq p < \infty$. This means that for any function u we have $u \in L^p(M)$ if and only $u \circ \Psi \in L^p(M_h)$ and $u \in H^{1,p}(M)$ if and only if $u \circ \Psi \in H^{1,p}(M_h)$ and that the norms of $L^p(M)$ and $L^p(M_h)$ as well as of $H^{1,p}(M)$ and $H^{1,p}(M_h)$ are equivalent. We shall have a closer look at the second property. The L^p and $H^{1,p}$-norms of a function $u \circ \Psi$ can be expressed using the metric distortion tensor. For any function $u \in L^p(M)$, we have

$$\|u \circ \Psi\|^p_{L^p(M_h)} = \int_M |u|^p \;\alpha_h\,\mathrm{d}vol. \tag{25}$$

For any point $x \in M$ such that $\Phi(x)$ is in the interior of a triangle of M_h, $\operatorname{grad} u$ at x and $\operatorname{grad}_{M_h}(u \circ \Psi)$ at $\Phi(x)$ satisfy

$$\operatorname{grad}_{M_h}(u \circ \Psi)(\Phi(x)) = \mathrm{d}\Psi^{\times}(\operatorname{grad} u(x)),$$

and the length of the gradient is given by

$$\left\|\operatorname{grad}_{M_h}(u \circ \Psi)(\Phi(x))\right\|^2_{\mathbb{R}^3} = \left\|\mathrm{d}\Psi^{\times}(\operatorname{grad} u(x))\right\|^2_{\mathbb{R}^3} = \left\langle A^{-1}\operatorname{grad} u(x), \operatorname{grad} u(x)\right\rangle_{\mathbb{R}^3}.$$

Then, for any function $u \in H^{1,p}(M)$, the $H^{1,p}$-norm of $u \circ \Psi$ satisfies

$$\|u \circ \Psi\|^p_{H^{1,p}(M_h)} = \|u\|^p_{L^p_h} + \int_M \left\langle A^{-1}\operatorname{grad} u, \operatorname{grad} u\right\rangle^{\frac{p}{2}}_{\mathbb{R}^3} \;\alpha_h\,\mathrm{d}vol. \tag{26}$$

The equivalence of the norms follows directly from the representations (25) and (26) and is summarized in the following lemma.

Lemma 6. *For every $u \in L^p(M)$, we have*

$$c_L \|u\|_{L^p}^p \leq \|u \circ \Psi\|_{L^p(M_h)}^p \leq C_L \|u\|_{L^p}^p \,, \tag{27}$$

where $c_L = \left\|\alpha_h^{-1}\right\|_{L^\infty}^{-1}$ *and* $C_L = \|\alpha_h\|_{L^\infty}$, *and for every $u \in H^{1,p}(M)$, we have*

$$c_H \|u\|_{H^{1,p}}^p \leq \|u \circ \Psi\|_{H^{1,p}(M_h)}^p \leq C_H \|u\|_{H^{1,p}}^p \,, \tag{28}$$

where $c_H = c_L + c_L \| A\|_\infty^{-\frac{p}{2}}$ *and* $C_H = C_L + C_L \left\|A^{-1}\right\|_\infty^{\frac{p}{2}}$.

5. Approximation of the Generalized Shape Operators

In this section, we derive error estimates for the approximation of the generalized shape operators of a smooth surface M by the generalized shape operators of a polyhedral surface M_h that is a normal graph over M. We begin with introducing appropriate measures for the distance between M and M_h as well as that between the generalized shape operators.

Since M_h is a normal graph over M, the *height* of M_h over M, given by $\sup_{x \in M_h} \delta_M(x)$, is a canonical measure for the spatial distance of M and M_h. This is confirmed by the following lemma, which states that the height agrees with the Hausdorff distance and the Fréchet distance of M and M_h.

Lemma 7. *Let M_h be a normal graph over a smooth surface M and let $\delta_H(M, M_h)$ and $\delta_F(M, M_h)$ denote the Hausdorff distance and the Fréchet distance of M and M_h. Then, we have*

$$\delta_H(M, M_h) = \delta_F(M, M_h) = \sup_{x \in M_h} \delta_M(x).$$

Proof. Since Ψ is a homeomorphism of M_h and M, we have

$$\delta_F(M, M_h) \leq \sup_{x \in M_h} \delta_M(x).$$

By definition, the Hausdorff distance of M and M_h is the maximum of $\sup_{x \in M_h} \delta_M(x)$ and $\sup_{x \in M} \inf_{y \in M_h} \|x - y\|_{\mathbb{R}^3}$. Hence, we have

$$\sup_{x \in M_h} \delta_M(x) \leq \delta_H(M, M_h).$$

Furthermore, the Hausdorff distance of two surfaces is smaller than their Fréchet distance,

$$\delta_H(M, M_h) \leq \delta_F(M, M_h).$$

The combination of the three inequalities proves the lemma. ∎

For our purposes, we prefer to use, instead of the height, the *relative height* of M_h over M:

$$\delta(M, M_h) = \sup_{x \in M_h} \delta_M(x)\, \kappa_{\max}(\Psi(x)), \tag{29}$$

which measures the spatial distance relative to the curvature of M. The resulting statements do not lose generality, since for any smooth surface M, the relative height of every normal graph M_h over M is bounded by a constant times its height,

$$\delta(M, M_h) \leq \|\kappa_{\max}\|_{L^\infty} \sup_{x \in M_h} \delta_M(x),$$

where $\|\kappa_{\max}\|_{L^\infty} < \infty$ since M is compact. The converse inequality does not hold in general, *e. g.*, the relative height of a two parallel planes vanishes whereas the height can be arbitrary large. The relative height has some more properties: since M_h is in the open reach(M)-tube round M, we have $\delta_M(x) < (\kappa_{\max}(\Psi(x)))^{-1}$ for all $x \in M_h$, which implies $\delta(M, M_h) \in [0, 1)$. Furthermore, $\delta(M, M_h)$ is invariant under scaling of M and M_h.

The lantern of Schwarz indicates that considering only the spatial distance does not suffice for our purposes, since it does not even imply convergence of the surface area. Under the assumption of convergence in the Hausdorff distance, Hildebrandt et al. (2006) proved that the convergence of the area form is equivalent to the convergence of the surface normal vectors. This motivate us to consider the distance of the normals of M_h and M. Explicitly, we use the value

$$\|N - N_h\|_{L^\infty},$$

where $N_h = N_{M_h} \circ \Phi$ is the pull-back to M of the piecewise constant normal N_{M_h} of the polyhedral surface M_h.

Definition 8 (ϵ-normal graph). *A polyhedral surface M_h is an ϵ-*normal graph *over a smooth surface M if M_h is a* normal graph *over M and satisfies $\delta(M, M_h) < \epsilon$ and $\|N - N_h\|_{L^\infty} < \sqrt{2}\epsilon$.*

To compare the generalized shape operators of M_h and M, we pull-back the operators from M_h to M, more explicitly, we consider the operators $\bar{\Sigma}_h$ and $\hat{\Sigma}_h$ given by

$$\begin{aligned} \bar{\Sigma}_h &: \mathcal{X}_{H^{1,1}}(M) \mapsto \mathbb{R}^3 \\ \bar{\Sigma}_h(X) &= \bar{\Sigma}_{M_h}(X \circ \Psi) \end{aligned}$$

and

$$\begin{aligned} \hat{\Sigma}_h &: \mathcal{X}_{H^{1,1}}(M) \mapsto \mathbb{R}^3 \\ \hat{\Sigma}_h(X) &= \hat{\Sigma}_{M_h}(X \circ \Psi). \end{aligned}$$

By construction, both operators, $\bar{\Sigma}_h$ and $\hat{\Sigma}_h$, are continuous operators and therefore elements of $L(\mathcal{X}_{H^{1,1}}(M), \mathbb{R}^3)$. Then, the operator norm $\| \ \|_{Op}$ of $L(\mathcal{X}_{H^{1,1}}(M), \mathbb{R}^3)$ measures the distance between $\bar{\Sigma}_h$ and $\bar{\Sigma}$ as well as that between $\hat{\Sigma}_h$ and $\bar{\Sigma}$.

Theorem 9. *Let M be a smooth surface in $\mathbb{R}^3$. Then, for every $\epsilon \in (0,1)$ there exists a constant C such that for every polyhedral surface M_h that is an ϵ-normal graph over M, the estimates*

$$\left\|\bar{\Sigma} - \bar{\Sigma}_h\right\|_{Op} \leq C\left(\delta(M, M_h) + \|N - N_h\|_{L^\infty}\right) \tag{30}$$

and

$$\left\|\hat{\Sigma} - \hat{\Sigma}_h\right\|_{Op} \leq C\left(\delta(M, M_h) + \|N - N_h\|_{L^\infty}\right) \tag{31}$$

hold. The constant C depends only on ϵ and converges to 2 for $\epsilon \to 0$.

Before we prove the theorem, we establish the following estimates for the divergence and the curl. For this we consider the pull-back of the divergence and the curl of M_h to M. These are the operators

$$\mathrm{div}_h : \mathcal{X}_{W^{1,1}}(M) \mapsto L^1(M) \qquad \text{and} \qquad \mathrm{curl}_h : \mathcal{X}_{W^{1,1}}(M) \mapsto L^1(M)$$

given by

$$\mathrm{div}_h(X)(x) = \mathrm{div}_{M_h}(X \circ \Psi)(\Phi(x)) \qquad \text{and} \qquad \mathrm{curl}_h(X)(x) = \mathrm{curl}_{M_h}(X \circ \Psi)(\Phi(x))$$

for almost all $x \in M$.

Lemma 10. *Let M be a smooth surface in $\mathbb{R}^3$ and $\epsilon \in (0,1)$. Then, for every polyhedral surface M_h that is an ϵ-normal graph over M, the estimates*

$$\|div - div_h\|_{Op} \leq \left(\|N_h - N\|_{L^\infty} + \frac{1}{1-\epsilon}\,\delta(M, M_h)\right)$$

and

$$\|curl - curl_h\|_{Op} \leq \left(\|N_h - N\|_{L^\infty} + \frac{1}{1-\epsilon}\,\delta(M, M_h)\right)$$

hold, where $\|\ \|_{Op}$ is the operator norm on $L(\mathcal{X}_{H^{1,1}}(M), L^1(M))$.

Proof. Let $X \in \mathcal{X}_{H^{1,1}}(M)$ be a vector field with $\|X\|_{H^{1,1}} = 1$, and let X_i be the components of X with respect to the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. Employing formula (2) and using the representation of $\mathrm{d}\Psi^\times$, described in equations (17) and (21), we get

$$\begin{aligned}
\mathrm{div}_h X &= \sum_{i=1}^{3} \left\langle \mathrm{d}\Psi^\times \,\mathrm{grad}\, X_i, e_i \right\rangle_{\mathbb{R}^3} \\
&= \sum_{i=1}^{3} \left\langle P^\times (Id + R)\,\mathrm{grad}\, X_i, e_i \right\rangle_{\mathbb{R}^3} \\
&= \mathrm{div} X - \sum_{i=1}^{3} \langle\, \mathrm{grad}\, X_i, N_h \rangle \, \langle N_h, e_i \rangle_{\mathbb{R}^3} + \sum_{i=1}^{3} \left\langle P^\times R\,\mathrm{grad}\, X_i, e_i \right\rangle_{\mathbb{R}^3}.
\end{aligned}$$

This yields an upper bound on the L^1-norm of the difference of the divergence operators

$$\|\operatorname{div} X - \operatorname{div}_h X\|_{L^1} \le \left\| \sum_{i=1}^{3} \langle \operatorname{grad} X_i, N_h \rangle \langle N_h, e_i \rangle_{\mathbb{R}^3} \right\|_{L^1} \tag{32}$$

$$+ \left\| \sum_{i=1}^{3} \left\langle P^{\times} R \operatorname{grad} X_i, e_i \right\rangle_{\mathbb{R}^3} \right\|_{L^1}$$

$$\le \left\| N_h^{\top} \right\|_{L^\infty} + \|R\|_\infty \le \|N_h - N\|_{L^\infty} + \|R\|_\infty \,,$$

where we use

$$\begin{aligned} \left\| N_h^{\top} \right\|^2_{\mathbb{R}^3} &= \|N_h - \langle N_h, N \rangle_{\mathbb{R}^3} N\|^2_{\mathbb{R}^3} = 1 + \langle N_h, N \rangle^2_{\mathbb{R}^3} - 2 \langle N_h, N \rangle^2_{\mathbb{R}^3} \\ &= 1 - \langle N_h, N \rangle^2_{\mathbb{R}^3} = (1 + \langle N_h, N \rangle_{\mathbb{R}^3})\,(1 - \langle N_h, N \rangle_{\mathbb{R}^3}) \\ &\le 2\,(1 - \langle N_h, N \rangle_{\mathbb{R}^3}) = \|N_h - N\|^2_{\mathbb{R}^3} \,. \end{aligned}$$

For any point $y \in M$ whose image $\Phi(y)$ is in the interior of a triangle of M_h, $R(y)$ is the symmetric linear map on $T_y M$ given by

$$R(y) = \sigma_M(\Phi(y))\, S(y)\, (Id - \sigma_M(\Phi(y))\, S(y))^{-1}, \tag{33}$$

see (18).

Now, let us consider the curl:

$$\begin{aligned} \operatorname{curl}_h X &= \sum_{i=1}^{3} \left\langle P^{\times} (Id + R) \operatorname{grad} X_i \times e_i, N_h \right\rangle_{\mathbb{R}^3} \\ &= \sum_{i=1}^{3} \left\langle \big(\operatorname{grad} X_i - \langle \operatorname{grad} X_i, N_h \rangle N_h + P^{\times} R \operatorname{grad} X_i \big) \times e_i, N_h \right\rangle_{\mathbb{R}^3} \\ &= \operatorname{curl} X + \sum_{i=1}^{3} \langle \operatorname{grad} X_i \times e_i, N_h - N \rangle_{\mathbb{R}^3} \\ &+ \sum_{i=1}^{3} \left\langle P^{\times} R \operatorname{grad} X_i \times e_i, N_h \right\rangle_{\mathbb{R}^3} \,, \end{aligned}$$

where we use

$$\langle \langle \operatorname{grad} X_i, N_h \rangle N_h \times e_i, N_h \rangle_{\mathbb{R}^3} = 0$$

in the last step. Then, we get the same upper bound on the L^1-norm of the difference of the curl operators as on the L^1-norm of the difference of the divergence operators:

$$\|\mathrm{curl}\, X - \mathrm{curl}_h X\|_{L^1} \leq \left\| \sum_{i=1}^{3} \langle\, \mathrm{grad}\, X_i \times e_i, N_h - N \rangle_{\mathbb{R}^3} \right\|_{L^1} \tag{34}$$
$$+ \left\| \sum_{i=1}^{3} \left\langle P^{\times} R\, \mathrm{grad}\, X_i \times e_i, N_h \right\rangle_{\mathbb{R}^3} \right\|_{L^1}$$
$$\leq \|N_h - N\|_{L^\infty} + \|R\|_\infty .$$

It remains to establish a bound on $\|R\|_\infty$. From (33) we can deduce that $R(y)$ has the same eigenvectors as the shape operator $S(y)$, and that the eigenvalues $\lambda_i(y)$ of $R(y)$ are given by

$$\lambda_i(y) = \frac{\sigma_M(\Phi(y))\, \kappa_i(y)}{1 - \sigma_M\, (\Phi(y))\kappa_i(y)}. \tag{35}$$

By the definition of $\delta(M, M_h)$, we have

$$\sigma_M(\Phi(y))\, \kappa_i(y) \leq \delta(M, M_h)$$

for all $y \in M$. Hence, we get

$$\|R\|_\infty \leq \frac{1}{1-\epsilon}\, \delta(M, M_h). \tag{36}$$

Combining (36) with (32) and (34), we see that the estimates

$$\|\mathrm{div} - \mathrm{div}_h\|_{Op} \leq \left(\|N_h - N\|_{L^\infty} + \frac{1}{1-\epsilon}\, \delta(M, M_h) \right)$$

and

$$\|\mathrm{curl} - \mathrm{curl}_h\|_{Op} \leq \left(\|N_h - N\|_{L^\infty} + \frac{1}{1-\epsilon}\, \delta(M, M_h) \right)$$

hold. This concludes the proof of the lemma. ∎

Now, we prove the theorem.

Proof of Theorem 9. Let $X \in \mathcal{X}_{H^{1,1}}$ be a vector field with $\|X\|_{H^{1,1}} = 1$. Then,

$$\left\|\bar{\Sigma} X - \bar{\Sigma}_h X\right\|_{\mathbb{R}^3} = \left\| \int_M (N\, \mathrm{div}\, X - N_h\, \mathrm{div}_h X\ \alpha_h) \mathrm{d}vol \right\|_{\mathbb{R}^3} \tag{37}$$
$$\leq \|(N - \alpha_h N_h) \mathrm{div}\, X\, \|_{L^1} + \|\alpha_h N_h (\mathrm{div}\, X - \mathrm{div}_h X)\, \|_{L^1}$$
$$\leq \|N - \alpha_h N_h\|_{L^\infty} \|\mathrm{div}\, X\, \|_{L^1} + \|\alpha_h\|_{L^\infty} \|\mathrm{div} - \mathrm{div}_h\|_{Op}$$
$$\leq \|1 - \alpha_h\|_{L^\infty} + \|N - N_h\|_{L^\infty} + \|\alpha_h\|_{L^\infty} \|\mathrm{div} - \mathrm{div}_h\|_{Op} ,$$

and similarly

$$\left\|\hat{\Sigma}X - \hat{\Sigma}_h X\right\|_{\mathbb{R}^3} = \left\|\int_M (N \operatorname{curl} X - N_h \operatorname{curl}_h X \ \alpha_h) \mathrm{d}vol\right\|_{\mathbb{R}^3} \tag{38}$$
$$\leq \|(N - \alpha_h N_h)\operatorname{curl} X\|_{L^1} + \|\alpha_h N_h (\operatorname{curl} X - \operatorname{curl}_h X)\|_{L^1}$$
$$\leq \|N - \alpha_h N_h\|_{L^\infty} \|\operatorname{curl} X\|_{L^1} + \|\alpha_h\|_{L^\infty} \|\operatorname{curl} - \operatorname{curl}_h\|_{Op}$$
$$\leq \|1 - \alpha_h\|_{L^\infty} + \|N - N_h\|_{L^\infty} + \|\alpha_h\|_{L^\infty} \|\operatorname{curl} - \operatorname{curl}_h\|_{Op} .$$

The ratio α_h of dvol and dvol_h is given by

$$\alpha_h(y) = \frac{1 - \sigma_M(\Phi(y))\, \frac{1}{2} H(y) + \sigma_M^2(\Phi(y))K(y)}{\langle N(y), N_h(y)\rangle_{\mathbb{R}^3}}, \tag{39}$$

see eq. (24). Our assumptions directly imply

$$\sigma_M(\Phi(y))\, \frac{1}{2} H(y) < \delta(M, M_h), \qquad \sigma_M^2(\Phi(y))K(y) < \epsilon\, \delta(M, M_h)$$

and

$$\langle N_h, N\rangle_{\mathbb{R}^3} = 1 - \frac{1}{2} \|N_h - N\|^2_{\mathbb{R}^3} .$$

This yields the upper bounds

$$\|1 - \alpha_h\|_{L^\infty} < \frac{2(1+\epsilon)\delta(M, M_h) + \|N_h - N\|^2_{\mathbb{R}^3}}{2 - \epsilon^2} \tag{40}$$

and

$$\|\alpha_h\|_{L^\infty} < 1 + \frac{2(1+\epsilon)\delta(M, M_h) + \|N_h - N\|^2_{\mathbb{R}^3}}{2 - \epsilon^2}. \tag{41}$$

Combining (37), Lemma 10, (40), and (41), we get

$$\left\|\bar{\Sigma} - \bar{\Sigma}_h\right\|_{Op} \leq (\|1 - \alpha_h\|_{L^\infty} + \|N - N_h\|_{L^\infty}) + \|\alpha_h\|_{L^\infty} \|\operatorname{div} - \operatorname{div}_h\|_{Op}$$
$$\leq C(\delta(M, M_h) + \|N - N_h\|_{L^\infty}),$$

where C is a constant that depends only on ϵ and satisfies $C \to 2$ for $\epsilon \to 0$. Similarly, using (38), Lemma 10, (40), and (41), we get

$$\left\|\hat{\Sigma} - \hat{\Sigma}_h\right\|_{Op} \leq (\|1 - \alpha_h\|_{L^\infty} + \|N - N_h\|_{L^\infty}) + \|\alpha_h\|_{L^\infty} \|\operatorname{curl} - \operatorname{curl}_h\|_{Op}$$
$$\leq C(\delta(M, M_h) + \|N - N_h\|_{L^\infty}).$$

This concludes the proof of the theorem. ∎

6. Pointwise Approximation of the Shape Operator

In this section, we derive estimates for the pointwise approximation of the shape operator of a smooth surface M in $\mathbb{R}^3$. They follow, as a corollary, from an estimate on the pointwise approximation of the tensor field $\bar{S}$. For sake of brevity, we restrict our considerations to the tensor field $\bar{S}$ and leave the tensor field $\hat{S}$ aside. Still, an analog statement to Theorem 13 holds for the approximation of the tensor field $\hat{S}$ as well.

r-local functions. The tool we use to obtain pointwise approximation estimates from the estimates in the operator norm are functions whose support becomes more and more localized while their L^1-norm remains constant and the growth of the $H^{1,1}$-norm is bounded. We define:

Definition 11. *Let $\mathcal{M}$ be a smooth or a polyhedral surface in $\mathbb{R}^3$, and let C_D be a positive constant. For any $x \in \mathcal{M}$ and $r \in \mathbb{R}^+$, we call a function $\varphi : \mathcal{M} \mapsto \mathbb{R}$ r-local at x (with respect to C_D) if the criteria*

(D1) $\varphi \in H^{1,1}(\mathcal{M})$,

(D2) $\varphi(y) \geq 0$ *for all* $y \in \mathcal{M}$,

(D3) $\varphi(y) = 0$ *for all* $y \in \mathcal{M}$ *with* $d_{\mathcal{M}}(x, y) \geq r$,

(D4) $\|\varphi\|_{L^1} = 1$, *and*

(D5) $|\varphi|_{H^{1,1}(\mathcal{M})} \leq \frac{C_D}{r}$

are satisfied.

A function that is r-local at $x \in M$ can be used to approximate the function value at x of a function f through the integral $\int_M f\,\varphi\,\mathrm{d}vol$. In this sense, r-local functions are approximations of the delta distribution.

Lemma 12. *Let $\varphi \in L^1(M)$ satisfy properties (D2), (D3), and (D4) of Definition 11 for some $x \in M$ and $r \in \mathbb{R}^+$, and let $f \in C^1(M)$. Then, the estimate*

$$\left| f(x) - \int_M f\,\varphi\,dvol \right| \leq \|\nabla f\|_{L^\infty}\, r \tag{42}$$

holds.

Proof. Since φ is non-negative and has unit L^1-norm, we have

$$\begin{aligned} \left| f(x) - \int_M f\,\varphi\,\mathrm{d}vol \right| &= \left| \int_M (f(x) - f)\,\varphi\,\mathrm{d}vol \right| \\ &\leq \sup_{y \in B_r(x)} |f(x) - f(y)|\,. \end{aligned}$$

For any y in the geodesic ball $B_r(x)$ around x, let γ be a (unit-speed parametrized) minimizing geodesic that connects x and y. Then

$$\begin{aligned} |f(x) - f(y)| &= \left| \int_\gamma g(\nabla f(\gamma(t)), \dot{\gamma}(t))\mathrm{d}t \right| \\ &\leq \|\nabla f\|_{L^\infty}\ \mathrm{length}(\gamma) \leq \|\nabla f\|_{L^\infty}\ r. \end{aligned}$$

This implies $\sup_{y \in B_r(x)} |f(x) - f(y)| \le \|\nabla f\|_{L^\infty} \, r$, which concludes the proof. ∎

Certain functions φ even exhibit a higher approximation order: there r-local functions φ that satisfy

$$\left| f(x) - \int_M f \, \varphi \, \mathrm{d}vol \right| \le C \, r^2, \tag{43}$$

where C depends on M and the second derivatives of f. We shall discuss the construction of such functions in Section 8.

Pointwise approximation of $\bar{S}$. Let ψ be r-local at the point $y \in M_h$. Testing the operator $\bar{\Sigma}_{M_h}$ with ψ generates a tensor $\bar{S}^\psi_{M_h}$ on $\mathbb{R}^3$ that has the components

$$(\bar{S}^\psi_{M_h})_{ij} = \left\langle e_i, \bar{\Sigma}_{M_h}(\psi \, e_j) \right\rangle_{\mathbb{R}^3}. \tag{44}$$

We shall show that $\bar{S}^\psi_{M_h}$ approximates $\bar{S}(\Psi(y))$. To measure the distance between $(1,1)$-tensors on $\mathbb{R}^3$, we use the operator norm

$$\|T\|_{\max} = \max_{v \in \mathbb{R}^3, \|v\|_{\mathbb{R}^3} = 1} \|Tv\|_{\mathbb{R}^3}$$

on the space of $(1,1)$-tensors on $\mathbb{R}^3$. Here, we use the subscript max instead of *op* to distinguish this norm from the operator norm on the space $L(\mathcal{X}_{H^{1,1}}(M), \mathbb{R}^3)$ used in the previous section.

Theorem 13. *Let M be a smooth surface in $\mathbb{R}^3$, and let $C_D \in \mathbb{R}^+$ and $\epsilon \in (0,1)$ be arbitrary but fixed. For every pair consisting of a polyhedral surface M_h that is an ϵ-normal graph over M and a function ψ that is r-local at a point $y \in M_h$ (with respect to C_D), the corresponding tensor $\bar{S}^\psi_{M_h}$ satisfies the estimate*

$$\left\| \bar{S}(x) - \bar{S}^\psi_{M_h} \right\|_{\max} \le C(r + (\delta(M, M_h) + \|N - N_h\|_{L^\infty})(\frac{1}{r} + 1)),$$

where $x = \Psi(y)$ is the orthogonal projection of y onto M. If $\psi \circ \Phi$ satisfies (43), the bound improves to

$$\left\| \bar{S}(x) - \bar{S}^\psi_{M_h} \right\|_{\max} \le C(r^2 + (\delta(M, M_h) + \|N - N_h\|_{L^\infty})(\frac{1}{r} + 1)).$$

The constant C depends only on M, ϵ, and C_D in both estimates.

Proof. Let $\varphi = \psi \circ \Phi$ be the pull-back to M of ψ, and let $i, j \in \{1, 2, 3\}$. Then,

$$\left(\bar{S}^\psi_{M_h} \right)_{ij} = \left\langle e_i, \bar{\Sigma}_h(\varphi \, e_j) \right\rangle_{\mathbb{R}^3}, \tag{45}$$

where $\bar{\Sigma}_h$ is the pull-back to M of $\bar{\Sigma}_{M_h}$. We set

$$\bar{\varphi} = \frac{\varphi}{\|\varphi\|_{L^1}}$$

and using (9) and (45), we get

$$\left|(\bar{S}(x) - \bar{S}^{\psi}_{M_h})_{ij}\right| = \left|(\bar{S}(x))_{ij} - \left\langle e_i, \bar{\Sigma}_h(\varphi\, e_j)\right\rangle_{\mathbb{R}^3}\right| \tag{46}$$

$$\leq \left|(\bar{S}(x))_{ij} - \int_M \bar{\varphi}\,(\bar{S})_{ij}\,\mathrm{d}vol\right| + \left|\left\langle e_i, \bar{\Sigma}(\bar{\varphi}\, e_j)\right\rangle_{\mathbb{R}^3} - \left\langle e_i, \bar{\Sigma}_h(\varphi\, e_j)\right\rangle_{\mathbb{R}^3}\right|.$$

In the following, we derive bounds for both summands of the right-hand side of (46). We start with the first summand. The function $\bar{\varphi}$ clearly satisfies *(D2)* and *(D4)* of Definition 11. Since the support of ψ is contained in the geodesic ball $B_r(y)$, the support of $\bar{\varphi}$ is contained in $B_{\|A\|_\infty r}(x)$, where A denotes metric distortion tensor; and it follows from (19) and $\tilde{(21)}$ that $\|A\|_\infty$ can be bounded by a constant C_A that depends only ϵ. Hence, $\bar{\varphi}$ satisfies property *(D3)* for the point x and the value $C_A r$. Then, Lemma 12 implies that there is a constant C depending on M and ϵ such that

$$\left|(\bar{S}(x))_{ij} - \int_M \bar{\varphi}\,(\bar{S})_{ij}\,\mathrm{d}vol\right| \leq C\,r$$

holds. If φ satisfies (43), then also $\bar{\varphi}$ satisfies (43) and the bound improves to $C\,r^2$.

To derive a bound on the second summand of the last row of (46), we split the summand in two parts:

$$\left|\left\langle e_i, \bar{\Sigma}(\bar{\varphi}\, e_j)\right\rangle_{\mathbb{R}^3} - \left\langle e_i, \bar{\Sigma}_h(\varphi\, e_j)\right\rangle_{\mathbb{R}^3}\right| \leq \left|\left\langle e_i, \bar{\Sigma}((\bar{\varphi}-\varphi)\ e_j)\right\rangle_{\mathbb{R}^3}\right| + \left|\left\langle e_i, (\bar{\Sigma}-\bar{\Sigma}_h)(\varphi\, e_j)\right\rangle_{\mathbb{R}^3}\right|.$$

The first part satisfies

$$\left|\left\langle e_i, \bar{\Sigma}((\bar{\varphi}-\varphi)\ e_j)\right\rangle_{\mathbb{R}^3}\right| = \left|\int_M (\bar{\varphi}-\varphi)\,(\bar{S})_{ij}\,\mathrm{d}vol\right|$$

$$= \left|(1-\|\varphi\|_{L^1})\int_M \bar{\varphi}\,(\bar{S})_{ij}\,\mathrm{d}vol\right| \leq \left|\|\varphi\|_{L^1} - 1\right| \left\|(\bar{S})_{ij}\right\|_{L^\infty},$$

and from the representation (39) of the volume distortion α_h, we can see that there is a constant C_α depending only on ϵ such that

$$\left|\|\varphi\|_{L^1} - 1\right| = \left|\int_M (1-\alpha_h)\,\varphi\,\mathrm{d}vol\right| \leq \left\|\frac{\alpha_h - 1}{\alpha_h}\right\|_{L^\infty} \leq C_\alpha \delta(M, M_h)$$

holds. To get a bound on the second part, we use Lemma 6:

$$\left|\left\langle e_i, (\bar{\Sigma}-\bar{\Sigma}_h)(\varphi\, e_j)\right\rangle_{\mathbb{R}^3}\right| \leq \|\varphi\|_{H^1} \left\|\bar{\Sigma}-\bar{\Sigma}_h\right\|_{Op}$$

$$\leq c_H^{-1}\, \|\psi\|_{H^1(M_h)} \left\|\bar{\Sigma}-\bar{\Sigma}_h\right\|_{Op} \leq c_H^{-1} \frac{C_D}{r} \left\|\bar{\Sigma}-\bar{\Sigma}_h\right\|_{Op}.$$

From the explicit representation of c_H in terms of the metric and volume distortion, we can deduce an upper bound on c_H^{-1} that depends only on ϵ. Furthermore, Theorem 9 provides the missing estimate for $\left\|\bar{\Sigma}-\bar{\Sigma}_h\right\|_{Op}$, which concludes the proof of the theorem. ■

Pointwise approximation of the shape operator. From the tensor $\bar{S}^{\psi}_{M_h}$, which approximates $\bar{S}(x)$, we can construct an approximation of $S(x)$. The principle of the construction is to remove the normal part of $\bar{S}^{\psi}_{M_h}$. In the case of a smooth surface, the definition of $\bar{S}(x)$ directly implies

$$S(x) = (Id - N(x)N(x)^T)\bar{S}(x)(Id - N(x)N(x)^T). \tag{47}$$

This motivates to define the tensor $S^{\psi}_{M_h}$ as

$$S^{\psi}_{M_h} = (Id - N_{M_h}(y)N_{M_h}(y)^T)\bar{S}^{\psi}_{M_h}(Id - N_{M_h}(y)N_{M_h}(y)^T). \tag{48}$$

Since the piecewise constant normal of the polyhedral surface is discontinuous at the edges and vertices, $N_{M_h}(y)$ is not well defined if y lies on an edge or a vertex of M_h. To get a well-defined tensor $S^{\psi}_{M_h}$, we specify what $N_{M_h}(y)$ means in this case: we set $N_{M_h}(y)$ to be the normalized sum of the normals of all triangles that are adjacent to the edge (respectively the vertex) on which y lies. An alternative would be to assign a triangle to each vertex and each edge and to use the normal of that triangle. For our purposes here, all such constructions yield the same asymptotic estimates. Now, we formulate our first estimate for the pointwise approximation of the shape operator of M.

Corollary 14. *Under the assumptions of Theorem 13, the tensor $S^{\psi}_{M_h}$ satisfies the estimate*

$$\left\| S(x) - S^{\psi}_{M_h} \right\|_{\max} \le C(r + (\delta(M, M_h) + \|N - N_h\|_{L^\infty})(\frac{1}{r} + 1)),$$

and if $\psi \circ \Phi$ satisfies (43),we get

$$\left\| S(x) - S^{\psi}_{M_h} \right\|_{\max} \le C(r^2 + (\delta(M, M_h) + \|N - N_h\|_{L^\infty})(\frac{1}{r} + 1)).$$

The constant C depends only on M, ϵ, and C_D in both estimates.

Proof. For simplicity of notation, we leave out the point, x or y, where the tensor and vector fields are evaluated, *i. e.*, we write S instead of $S(x)$ and N and N_h instead of $N(x)$ and $N_h(x)$. Using the pull-back N_h to M of the normal N_{M_h} of M_h and equations (47) and (48), we get

$$\begin{aligned}
&\left\| S(x) - S^{\psi}_{M_h}(y) \right\|_{\max} \\
&= \left\| (Id - NN^T)\bar{S}(Id - NN^T) - (Id - N_hN_h^T)\bar{S}^{\psi}_{M_h}(Id - N_hN_h^T) \right\|_{\max} \\
&= \| (N_hN_h^T - NN^T)\bar{S}(Id - NN^T) + (Id - N_hN_h^T)\bar{S}(N_hN_h^T - NN^T) \\
&+ (Id - N_hN_h^T)(\bar{S}^{\psi}_{M_h} - \bar{S})(N_hN_h^T - NN^T) + (Id - N_hN_h^T)(\bar{S} - \bar{S}^{\psi}_{M_h})(Id - NN^T) \|_{\max} \\
&\le 2 \left\| N_hN_h^T - NN^T \right\|_{\max} C_{\bar{S}} + (1 + \left\| N_hN_h^T - NN^T \right\|_{\max}) \left\| \bar{S}(x) - \bar{S}^{\psi}_{M_h} \right\|_{\max}.
\end{aligned}$$

Combining this with Theorem 13 and the estimate

$$\begin{aligned}\left\|N_h N_h^T - N N^T\right\|_{\max} &\leq \left\|(N_h - N) N^T\right\|_{\max} + \left\|N_h (N_h^T - N^T)\right\|_{\max} \\ &\leq 2 \left\|(N_h - N)\right\|_{L^\infty}\end{aligned}$$

proves the corollary. ■

The estimates in Theorem 13 and Corollary 14 depend on r and $\delta(M, M_h) + \|N - N_h\|_{L^\infty}$, and both quantities are independent. The following corollary shows how to choose r to get the optimal approximation order in $\delta(M, M_h) + \|N - N_h\|_{L^\infty}$.

Corollary 15. *Under the assumptions of Theorem 13 and the additional assumption that* $r = \sqrt{\delta(M, M_h) + \|N - N_h\|_{L^\infty}}$, *we get the estimate*

$$\left\|S(x) - S_{M_h}^{\psi}\right\|_{\max} \leq C \sqrt{\delta(M, M_h) + \|N - N_h\|_{L^\infty}},$$

and if $\psi \circ \Phi$ *satisfies (43) and* $r = (\delta(M, M_h) + \|N - N_h\|_{L^\infty})^{\frac{1}{3}}$, *we get*

$$\left\|S(x) - S_{M_h}^{\psi}\right\|_{\max} \leq C \left(\delta(M, M_h) + \|N - N_h\|_{L^\infty}\right)^{\frac{2}{3}}.$$

The constant C *depends only on* M, ϵ, *and* C_D *in both estimates.*

Proof. The corollary immediately follows from Corollary 14 and the assumption that $r = \sqrt{\delta(M, M_h) + \|N - N_h\|_{L^\infty}}$ respectively $r = (\delta(M, M_h) + \|N - N_h\|_{L^\infty})^{\frac{1}{3}}$. ■

Uniform approximation of S. Let us consider a family $\{\psi_y\}_{y \in M_h}$ of functions on M_h such that for every point $y \in M_h$ the function ψ_y is r-local at y with respect to the same constant C_D. Then $y \mapsto S_{M_h}^{\psi_y}$ is a tensor field on $M_h \times \mathbb{R}^3$ and we can show that the pointwise approximation estimates hold uniformly on M.

Corollary 16. *Let* M *be a smooth surface in* $\mathbb{R}^3$, *and let* $C_D \in \mathbb{R}^+$ *and* $\epsilon \in (0, 1)$ *be arbitrary but fixed. For every pair consisting of a polyhedral surface* M_h *that is an* ϵ*-normal graph over* M *and a family* $\{\psi_y\}_{y \in M_h}$ *of functions on* M_h *such that for every point* $y \in M_h$ *the function* ψ_y *is* r*-local at* y *(with respect to* C_D*), the corresponding tensor field* $y \mapsto S_{M_h}^{\psi_y}$ *satisfies the estimate*

$$\sup_{y \in M_h} \left\|S(x) - S_{M_h}^{\psi_y}\right\|_{\max} \leq C\left(r + (\delta(M, M_h) + \|N - N_h\|_{L^\infty})\left(\frac{1}{r} + 1\right)\right), \quad (49)$$

where $x = \Psi(y)$ *is the orthogonal projection of* y *onto* M *and the constant* C *depends only on* M, ϵ, *and* C_D.

Proof. From Corollary 14 we know that the estimate

$$\left\|S(x) - S_{M_h}^{\psi_y}\right\|_{\max} \leq C\left(r + (\delta(M, M_h) + \|N - N_h\|_{L^\infty})\left(\frac{1}{r} + 1\right)\right)$$

holds for all $y \in M_h$ with the same constant C. Hence, the supremum over all $y \in M_h$ satisfies the estimate as well. ■

Approximation of curvatures. Since the estimates of Corollaries 14, 15, and 16 hold for every component of the shape operator, they directly imply analog estimates for the mean and Gaussian curvature, as well as the principal curvatures and directions.

7. Inscribed Polyhedral Surfaces

In this section, we specialize the approximation estimates for the shape operator to polyhedral surfaces whose vertices lie on the smooth surface M, so-called inscribed polyhedral surfaces.

Definition 17. *We call a polyhedral surface M_h* inscribed *to a smooth surface M if M_h is a normal graph over M and all vertices of M_h are on the surface M.*

For inscribed polyhedral surfaces, the relative height, $\delta(M, M_h)$, and the approximation of the normals, $\|N - N_h\|_{L^\infty}$, can be bounded above in terms of the mesh size h, the shape regularity ρ, and properties of M, compare Nédélec (1976); Amenta et al. (2000); Morvan and Thibert (2004); Morvan (2008). We summarize this in the following lemma.

Lemma 18. *Let M be a smooth surface in $\mathbb{R}^3$. Then, there exists an $h_0 \in \mathbb{R}^+$ such that for every polyhedral surface M_h that is inscribed to M and satisfies $h < h_0$ the inequalities*

$$\delta(M, M_h) \leq C_H\, h^2 \tag{50}$$

and

$$\|N - N_h\|_{L^\infty} \leq C_N\, h \tag{51}$$

hold, where C_H and C_N depend only on M and the shape regularity ρ of M_h.

By restricting our considerations to inscribed polyhedral surfaces and using Lemma 18, we can obtain approximation estimates that depend on h instead of $\delta(M, M_h)$ and $\|N - N_h\|_{L^\infty}$.

Lemma 19. *Let M be a smooth surface in $\mathbb{R}^3$. Then, there exists an $h_0 \in \mathbb{R}^+$ such that for every polyhedral surface M_h that is inscribed to M and satisfies $h < h_0$ the inequalities*

$$\left\|\bar{\Sigma} - \bar{\Sigma}_h\right\|_{Op} \leq C\, h \qquad \text{and} \qquad \left\|\hat{\Sigma} - \hat{\Sigma}_h\right\|_{Op} \leq C\, h$$

hold, where C depends only on M, h_0 and the shape regularity of M_h.

Proof. To prove the lemma, we combine the estimates (30) and (31) of Theorem 9 with (50) and (51) and choose h_0 and C accordingly. ■

Furthermore, specializing Corollary 14 to inscribed meshes yields estimates on the pointwise approximation that depend on the mesh size h.

Lemma 20. *Let M be a smooth surface in $\mathbb{R}^3$, and let $C_D \in \mathbb{R}^+$ be arbitrary but fixed. Then, there exists an $h_0 \in \mathbb{R}^+$ such that for every pair consisting of a polyhedral surface M_h that is inscribed to M and satisfies $h < h_0$ and a function ψ that is r-local at a point $y \in M_h$ (with respect to C_D) with $r = \sqrt{h}$, the corresponding tensor $S^{\psi}_{M_h}$ satisfies the estimate*

$$\left\| S(x) - S^{\psi}_{M_h} \right\|_{\max} \leq C\sqrt{h} \tag{52}$$

where $x = \Psi(y)$ is the orthogonal projection of y onto M. If $\psi \circ \Phi$ satisfies (43) and $r = h^{\frac{1}{3}}$, the error bound improves to

$$\left\| S(x) - S^{\psi}_{M_h} \right\|_{\max} \leq C\, h^{\frac{2}{3}}. \tag{53}$$

The constant C depends only on M, h_0, ρ, and C_D in both estimates.

Proof. The lemma immediately follows from combining Corollary 14 with Lemma 18. ■

8. Examples of r-local Functions

In this section, we discuss the construction of r-local functions first on smooth and then on polyhedral surfaces. By a family of r-local functions at $x \in M$, we mean a family $(\phi_r)_{(0,\rho)}$ such that for all $r \in (0, \rho)$, ϕ_r is r-local at x with respect to fixed a constant C_D. We discuss two possible constructions: first, we consider a family of r-local functions on $\mathbb{R}^2$ and then use the Riemannian exponential map to construct a family of r-local functions on M, and, second, we construct a specific family of r-local functions on M based on the extrinsic distance of points in $\mathbb{R}^3$.

Let $\phi \in H^{1,1}(\mathbb{R}^2)$ be a non-negative function that vanishes in the complement of the open unit ball in $\mathbb{R}^2$ and satisfies $\|\phi\|_{L^1(\mathbb{R}^2)} = 1$. Then, ϕ_r defined by

$$\phi_r(\cdot) = \frac{1}{r^2}\phi(\frac{\cdot}{r})$$

is a family of r-local functions on $\mathbb{R}^2$, and the constant C_D assumes the value $\|\phi\|_{H^{1,1}(\mathbb{R}^2)}$. Since the surface M is compact, the injectivity radius $i(M)$ of M is a strictly positive number. For a point $x \in M$ and an $r \in \mathbb{R}^+$, let $B_r(x)$ be the open geodesic ball around x in M, and let $B_r(0)$ denote the open ball of radius r around the origin 0 in T_xM. The Riemannian exponential map at the point $x \in M$,

$$\exp : B_{i(M)}(0) \subset T_xM \mapsto M,$$

is a diffeomorphism of $B_{i(M)}(0)$ and $\exp(B_{i(M)}(0)) = B_{i(M)}(x)$. Let $\rho \in \mathbb{R}^+$ be strictly smaller than $i(M)$. Then, the family $(\varphi_r)_{r\in(0,\rho)}$ given by

$$\varphi_r = \left\|\phi_r \circ \exp^{-1}\right\|^{-1}_{L^1(M)} \phi_r \circ \exp^{-1}$$

is a family of r-local functions at x. The properties *(D2)* and *(D4)* of Definition 11 are clearly satisfied, and *(D3)* holds since exp is a radial isometry. Since exp is a diffeomorphism on $B_{i(M)}(0)$, *(D1)* follows from properties of Sobolev spaces under smooth coordinate transformations, see (Adams, 1975, Theorem 3.35). To show that *(D5)* holds, we use $\exp : B_{i(M)}(0) \mapsto M$ as a parametrization of M around x. Since the support of φ_r is is contained in $\exp(B_{i(M)}(0))$ for all $r \in (0, \rho)$, we can calculate $\|\varphi_r\|_{H^{1,1}}$ using only the chart exp. Analog to our discussion on the metric distortion introduced by the cone metric of a polyhedral surface (see eq. (22)), we can represent the metric distortion induced by exp through a metric distortion tensor $A_{\exp}$ and the distortion of the volume form by a function $\alpha_{\exp} = \sqrt{\det(A_{\exp})}$. On the compact set $\overline{B_\rho(0)}$, $\alpha_{\exp}$ and the eigenvalues of $A_{\exp}$ are bounded above and below, and since exp is a diffeomorphism, the lower bounds are strictly larger than zero. Then, there are constants c and C such that

$$c\,\|u\|_{L^1(M)} \leq \left\|u \circ \exp^{-1}\right\|_{L^1(\mathbb{R}^2)} \leq C\,\|u\|_{L^1(M)} \tag{54}$$

holds for all $u \in L^1(M)$ whose support is contained in $\overline{B_\rho(0)}$, and there are constants $\tilde{c}$ and $\tilde{C}$ such that

$$\tilde{c}\,\|u\|_{H^{1,1}(M)} \leq \left\|u \circ \exp^{-1}\right\|_{H^{1,1}(\mathbb{R}^2)} \leq \tilde{C}\,\|u\|_{H^{1,1}(M)} \tag{55}$$

holds for all all $u \in H^{1,1}(M)$ whose support is contained in $\overline{B_\rho(x)}$. Because the support of φ_r is contained in the compact set $\overline{B_\rho(x)}$, we have

$$\left\|\phi_r \circ \exp^{-1}\right\|_{H^{1,1}(M)} \leq \tilde{C}\,\|\phi_r\|_{H^{1,1}(\mathbb{R}^2)} \tag{56}$$

and

$$\left\|\phi_r \circ \exp^{-1}\right\|_{L^1(M)} \geq c\,\|\phi_r\|_{L^1(\mathbb{R}^2)} = c \tag{57}$$

for all $r \in (0, \rho)$. It follows that the estimate

$$\|\varphi_r\|_{H^{1,1}(M)} \leq \frac{\tilde{C}\,\|\phi\|_{H^{1,1}(\mathbb{R}^2)}}{c}\frac{1}{r} \tag{58}$$

is satisfied for all $r \in (0, \rho)$. This means *(D5)* holds as well.

Geodesic hat functions. As an example of this construction of r-local functions, let us consider the function $\phi(\cdot) = \frac{3}{\pi}\max\{0, 1 - \|\cdot\|_{\mathbb{R}^2}\}$ on $\mathbb{R}^2$ (resp. on T_xM). We call the corresponding functions φ_r on M *geodesic hat functions*, since they decay linearly with the geodesic distance to x. Explicitly, φ_r is given by

$$\varphi_r = \frac{\tilde{\varphi}_r}{\|\tilde{\varphi}_r\|_{L^1}}, \text{ where } \tilde{\varphi}_r(y) = \max\{0, 1 - \frac{d_M(x,y)}{r}\}. \tag{59}$$

The function φ_r is an example of a function that satisfies the estimate (43).

Extrinsic hat functions. To keep computations simple, one can employ the extrinsic distance of points in $\mathbb{R}^3$ instead of the geodesic distance. The *extrinsic hat function* is defined as

$$\psi_r(y) = \frac{\tilde{\psi}_r(y)}{\left\|\tilde{\psi}_r\right\|_{L^1(M)}}, \text{ where } \tilde{\psi}_r(y) = \max\{1 - \frac{\|x-y\|_{\mathbb{R}^3}}{r}, 0\}. \quad (60)$$

As above, we focus on the properties of ψ_r for small values of r, and, therefore, we fix a small ρ consider only ψ_r with $r \in (0, \rho)$. Then, the ψ_rs satisfy the properties of r-local functions, except that we need to modify property *(D3)*: the support of ψ_r is not contained in $B_r(x)$ but there is a constant C depending only on M such that $\operatorname{supp}(\psi_r) \subset B_{Cr}(x)$. The approximation estimates derived in Section 6 hold for these functions as well, and we can show that the ψ_rs satisfy property (43).

Polyhedral surfaces. The two constructions (59) and (60) can be directly transferred to polyhedral surfaces, where the geodesic distance on M is replaced by the geodesic distance on M_h. On polyhedral surfaces our focus is not on the construction of r-local functions with arbitrary small r, but with certain values like $r = h^{\frac{1}{2}}$ or $r = (\delta(M, M_h) + \|N - N_h\|_{L^\infty})^{\frac{1}{2}}$. Then, r is large compared to h, resp. $\delta(M, M_h) + \|N - N_h\|_{L^\infty}$, and one can show that the pullback to M of a function on M_h constructed as described above still satisfies (43). It is convenient to work with continuous and piecewise linear functions on M_h, *e. g.* with the interpolants of a function. The gradient of such a function is constant in each triangle and one can evaluate the generalized shape operators by summing over the triangles in the support of the function. The functions used in the experiments, see (61), are an example of such a construction.

9. Experiments

In this Section, we show the results of three experiments concerning the error and convergence rate of the approximation of the shape operator. In the first example, we approximate the tensors $\bar{S}(x)$ and $\hat{S}(x)$ at a point x on the unit sphere in $\mathbb{R}^3$ using inscribed polyhedral surfaces with decreasing mesh size h. On each polyhedral surface M_h we consider two functions, ψ and ψ^*. Both functions are continuous and linear on the triangles and hence are determined by their values at the vertices of M_h. At any vertex v of M_h, the functions take the values

$$\psi(v) = \max\{1 - \frac{\|x-v\|_{\mathbb{R}^3}}{\sqrt{h}}, 0\} \quad (61)$$

and

$$\psi^*(v) = \max\{1 - \frac{\left\|x + \frac{\sqrt{h}}{20}e - v\right\|_{\mathbb{R}^3}}{\sqrt{h}}, 0\}, \quad (62)$$

where h is the mesh size of M_h and e is a fixed unit vector in $\mathbb{R}^3$.

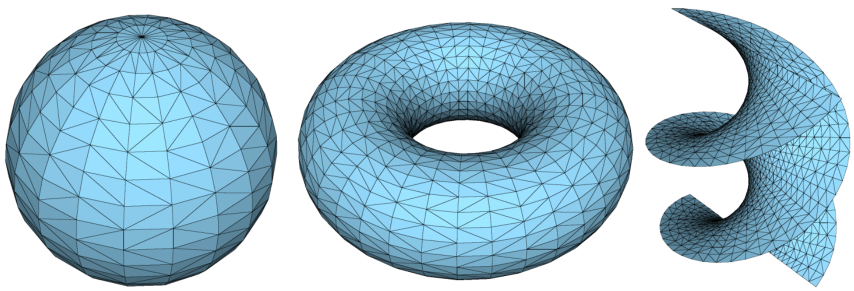

Figure 3: Some of the surfaces used for the experiments are shown.

Using ψ and ψ^*, we construct the tensors $\bar{S}^{\psi}_{M_h}$ and $\hat{S}^{\psi}_{M_h}$ with components

$$(\bar{S}^{\psi}_{M_h})_{ij} = \frac{\left\langle e_i, \bar{\Sigma}_{M_h}(\psi\, e_j)\right\rangle_{\mathbb{R}^3}}{\|\psi\|_{L^1(M_h)}} \text{ and } (\hat{S}^{\psi}_{M_h})_{ij} = \frac{\left\langle e_i, \hat{\Sigma}_{M_h}(\psi\, e_j)\right\rangle_{\mathbb{R}^3}}{\|\psi\|_{L^1(M_h)}} \tag{63}$$

and the tensors $\bar{S}^{\psi^*}_{M_h}$ and $\hat{S}^{\psi^*}_{M_h}$ with components

$$(\bar{S}^{\psi^*}_{M_h})_{ij} = \frac{\left\langle e_i, \bar{\Sigma}_{M_h}(\psi^*\, e_j)\right\rangle_{\mathbb{R}^3}}{\|\psi^*\|_{L^1(M_h)}} \text{ and } (\hat{S}^{\psi^*}_{M_h})_{ij} = \frac{\left\langle e_i, \hat{\Sigma}_{M_h}(\psi^*\, e_j)\right\rangle_{\mathbb{R}^3}}{\|\psi^*\|_{L^1(M_h)}}. \tag{64}$$

Table 1 lists the approximation errors of the four tensors (measured in the norm $\|\ \|_{\max}$) for inscribed polyhedral surfaces with decreasing mesh size h. In addition, the table shows the experimental order of convergence. Let e_{h_i} and $e_{h_{i+1}}$ be the approximation errors of some quantity for the decreasing mesh sizes h_i and h_{i+1}. Then, the *experimental order of convergence (eoc)* of the quantity is defined as

$$\operatorname{eoc}(h_i, h_{i+1}) = \log \frac{e_{h_i}}{e_{h_{i+1}}} \left(\log \frac{h_i}{h_{i+1}} \right)^{-1}.$$

All four approximations converge, but the order of convergence differs depending on which function, ψ or ψ^*, we use. The experimental order of convergence of $\bar{S}^{\psi^*}_{M_h}$ and $\hat{S}^{\psi^*}_{M_h}$ is $\frac{1}{2}$, which confirms the sharpness of our estimates. However, the function ψ leads to an eoc of 1 in our experiments, even if we add normal noise (of order h^2) and even stronger tangential noise to the vertex positions of the polyhedral surfaces. When replacing $\sqrt{h}$ by $h^{\frac{1}{3}}$ in (61), we got the expected eoc of $\frac{2}{3}$.

The second example concerns the approximation of the classic shape operator of a smooth surface. We consider a point x on a torus of revolution (with radii 1 and 2) and polyhedral surfaces that are inscribed to the torus. On each polyhedral surface we use the function ψ, see (61), to compute the tensors $\bar{S}^{\psi}_{M_h}$ and $\hat{S}^{\psi}_{M_h}$, similar to the first example. In this example, we additionally construct

h	$\|\bar{S}(x) - \bar{S}^{\psi}_{M_h}\|$	eoc	$\|\bar{S}(x) - \bar{S}^{\psi^*}_{M_h}\|$	eoc
0.0744108	0.0684689	–	0.0828663	–
0.0304109	0.0275034	1.00	0.0409642	0.79
0.0102627	0.0092561	1.00	0.0197543	0.67
0.0030374	0.0027360	1.00	0.0096611	0.59
0.0008309	0.0007480	1.00	0.0048014	0.54
0.0002176	0.0001959	1.00	0.0023992	0.52
0.0000557	0.0000501	1.00	0.0012003	0.51
0.0000146	0.0000132	1.00	0.0006122	0.50

h	$\|\hat{S}(x) - \hat{S}^{\psi}_{M_h}\|$	eoc	$\|\hat{S}(x) - \hat{S}^{\psi^*}_{M_h}\|$	eoc
0.0744108	0.0114370	–	0.0181622	–
0.0304109	0.0045876	1.00	0.0101977	0.65
0.0102627	0.0015431	1.00	0.0055643	0.56
0.0030374	0.0004560	1.00	0.0029508	0.52
0.0008309	0.0001247	1.00	0.0015321	0.51
0.0002176	0.0000326	1.00	0.0007826	0.50
0.0000557	8.36×10^{-6}	1.00	0.0003958	0.50
0.0000146	2.20×10^{-6}	1.00	0.0002030	0.50

Table 1: Approximations of the tensors $\bar{S}(x)$ and $\hat{S}(x)$ at a point x on the unit sphere are analyzed, the approximation error and experimental rate of convergence are shown.

approximations of the normal of M at x: for every polyhedral surface M_h we set

$$N^{\tilde{\psi}}_{M_h} = \frac{1}{\left\|\int_{M_h} N_{M_h}\, \tilde{\psi}\, \mathrm{d}vol_h\right\|_{\mathbb{R}^3}} \int_{M_h} N_{M_h}\, \tilde{\psi}\, \mathrm{d}vol_h,$$

where $\tilde{\psi}$ is the continuous and piecewise linear function on M_h that takes the values

$$\tilde{\psi}(v) = \max\{1 - \frac{\|x - v\|_{\mathbb{R}^3}}{2h}, 0\} \tag{65}$$

at the vertices v. We use $N^{\tilde{\psi}}_{M_h}$ instead of evaluating $N_{M_h}(\Phi(x))$ to avoid the need to compute the point $\Phi(x)$. Using the estimated normal, we construct the following two approximations of $S(x)$: the first is defined, analog to (48), by

$$S^{\psi}_{M_h} = (Id - N^{\tilde{\psi}}_{M_h} {N^{\tilde{\psi}}_{M_h}}^T)\bar{S}^{\psi}_{M_h}(Id - N^{\tilde{\psi}}_{M_h} {N^{\tilde{\psi}}_{M_h}}^T), \tag{66}$$

and the second (denoted by a calligraphic letter) is given by

$$\mathcal{S}^{\psi}_{M_h} = \hat{S}^{\psi}_{M_h}\, C_{N^{\tilde{\psi}}_{M_h}}, \tag{67}$$

where

$$C_{N^{\tilde{\psi}}_{M_h}} = \begin{pmatrix} 0 & (N^{\tilde{\psi}}_{M_h})_3 & -(N^{\tilde{\psi}}_{M_h})_2 \\ -(N^{\tilde{\psi}}_{M_h})_3 & 0 & (N^{\tilde{\psi}}_{M_h})_1 \\ (N^{\tilde{\psi}}_{M_h})_2 & -(N^{\tilde{\psi}}_{M_h})_1 & 0 \end{pmatrix}$$

h	$\|\bar{S}(x) - \bar{S}^{\psi}_{M_h}\|$	eoc	$\|S(x) - S^{\psi}_{M_h}\|$	eoc
0.0442741	0.0175626	–	0.0155477	–
0.0241741	0.0095546	1.00	0.0084655	1.00
0.0102634	0.0040565	1.00	0.0035977	1.00
0.0035994	0.0014209	1.00	0.0012606	1.00
0.0010920	0.0004308	1.00	0.0003822	1.00
0.0003030	0.0001195	1.00	0.0001060	1.00
0.0000800	0.0000315	1.00	0.0000280	1.00
0.0000206	8.11×10^{-6}	1.00	7.19×10^{-6}	1.00

h	$\|\hat{S}(x) - \hat{S}^{\psi}_{M_h}\|$	eoc	$\|S(x) - \mathcal{S}^{\psi}_{M_h}\|$	eoc
0.0442741	0.0027184	–	0.0036529	–
0.0241741	0.0014769	1.00	0.0020020	0.99
0.0102634	0.0006264	1.00	0.0008539	0.99
0.0035994	0.0002193	1.00	0.0002998	1.00
0.0010920	0.0000665	1.00	0.0000910	1.00
0.0003030	0.0000184	1.00	0.0000252	1.00
0.0000800	4.87×10^{-6}	1.00	6.66×10^{-6}	1.00
0.0000206	1.25×10^{-6}	1.00	1.71×10^{-6}	1.00

Table 2: The table shows the approximation error and experimental rate of convergence of approximations of the shape operator at a point x on the torus of revolution.

is the matrix representation of the cross product with the vector $-N^{\tilde{\psi}}_{M_h}$. In our experiments, both tensors, $S^{\psi}_{M_h}$ and $\mathcal{S}^{\psi}_{M_h}$, converge to $S(x)$ with the same order as $\bar{S}^{\psi}_{M_h}$ and $\hat{S}^{\psi}_{M_h}$ converge to $\bar{S}(x)$ and $\hat{S}(x)$, see Table 2.

The third example concerns the approximation of the shape operator from polyhedral surfaces that are corrupted by noise and are not inscribed anymore. We consider the shape operator at a point x on the helicoid in $\mathbb{R}^3$ and compute the tensor $S^{\psi}_{M_h}$ (in the same way as in the second example) first on inscribed polyhedral surfaces. Then, we disturb the polyhedral surfaces, by adding random noise of order h^2 to the vertex positions, and compute the operator again. We denote the operator computed from the distorted surface by $S^{\psi}_{M_h,noise}$. In our experiments, we found the same order of convergence for both operators, see Table 3. In addition, the table lists approximation errors and eoc for the tensors $S^{\tilde{\psi}}_{M_h}$ and $S^{\tilde{\psi}}_{M_h,noise}$ which were computed on the same surfaces but using the function $\tilde{\psi}$, see (65), instead of ψ. The main difference between $S^{\psi}_{M_h}$ and $S^{\tilde{\psi}}_{M_h}$ is that the regions on the surfaces that is used to compute $S^{\psi}_{M_h}$ and $S^{\psi}_{M_h,noise}$ is larger then the regions used to estimate $S^{\tilde{\psi}}_{M_h}$ and $S^{\tilde{\psi}}_{M_h,noise}$; the support of ψ is of order $\sqrt{h}$ and the support of $\tilde{\psi}$ is of order h. When computed from the surface without noise, the tensor $S^{\tilde{\psi}}_{M_h}$ converges to $S(x)$ (even with order 2 in our experiments), but when computed from the corrupted surface, the tensor $S^{\tilde{\psi}}_{M_h}$ does not converge anymore.

h	$\|S(x) - S^{\psi}_{M_h}\|$	eoc	$\|S(x) - S^{\psi}_{M_h,noise}\|$	eoc
0.0385465	0.0009560	–	0.0014489	–
0.0260549	0.0006113	1.10	0.0014966	−0.08
0.0141667	0.0003216	1.10	0.0004560	2.00
0.0060853	0.0001362	1.00	0.0001497	1.30
0.0021316	0.0000476	1.00	0.0000687	0.74
0.0006460	0.0000144	1.00	0.0000194	1.10
0.0001791	3.99×10^{-6}	1.00	5.22×10^{-6}	1.00

h	$\|S(x) - S^{\tilde{\psi}}_{M_h}\|$	eoc	$\|S(x) - S^{\tilde{\psi}}_{M_h,noise}\|$	eoc
0.0385465	0.0002335	–	0.0047959	–
0.0260549	0.0000971	2.20	0.0165850	−3.20
0.0141667	0.0000249	2.20	0.0062470	1.60
0.0060853	4.01×10^{-6}	2.20	0.0213477	−1.50
0.0021316	8.96×10^{-7}	1.40	0.0053238	1.30
0.0006460	7.97×10^{-8}	2.00	0.0132693	−0.76
0.0001791	6.00×10^{-9}	2.00	0.0204679	−0.34

Table 3: The table shows the error and the experimental rate of convergence for the approximation of the shape operator at a point on the helicoid. The columns on the right show results computed from polyhedral surfaces that are corrupted with noise.

10. Conclusion

We have presented generalized shape operators that are linear operators on function spaces of weakly differentiable vector fields and can be defined for smooth and polyhedral surfaces. We have proved error estimates for approximation of the generalized shape operators in the operator norm and for pointwise approximation of the classic shape operator of smooth surfaces from polyhedral surfaces. Our estimates are confirmed by numerical experiments.

Though based on different mathematical techniques, for applications our generalized shape operators can be used in a similar manner as the generalized curvatures or discrete curvatures. In applications, discrete curvatures provide good results for the pointwise approximation of curvatures, *e. g.* to compute principal curvatures or principal curvature directions. Our pointwise approximation estimates throw some light upon the question why this works and provide a theoretical justification for such usage.

As an extension of this work, we plan to apply the presented technique to obtain pointwise approximation estimates for the discrete Laplace–Beltrami operator of polyhedral surfaces. Another open question is motivated by our experiments with parametrized surfaces and inscribed polyhedral surfaces. We observed an experimental order of convergence of h for the pointwise approximation of the shape operator from generalized shape operators tested with functions that satisfy (43), where an order of $\sqrt{h}$ was expected. We got the same convergence order even when noise of order h^2 was added to the vertex positions of the polyhedral surface. This leads to the question whether one can prove an $\mathcal{O}(h)$ bound on the approximation error.

Acknowledgements. This work was supported by the DFG Research Center MATHEON "Mathematics for Key Technologies" in Berlin. We would like to thank the anonymous reviewers for valuable comments and feedback.

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